COSMIC MICROWAVE BACKGROUND ANISOTROPIES UP TO SECOND ORDER

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1. Preamble

These lecture notes present the computation of the full system of Boltzmann equations describing the evolution of the photon, baryon and cold dark matter fluids up to second order in perturbation theory, as recently studied in Refs. [1, 2]. These equations allow to follow the time evolution of the cosmic microwave background anisotropies at all angular scales from the early epoch, when the cosmological perturbations were generated, to the present, through the recombination era. The inclusion of second-order contributions is mandatory when one is interested in studying possible deviations from Gaussianity of cosmological perturbations, either of primordial (e.g. inflationary) origin or due to their subsequent evolution. Most of the emphasis in these lectures notes will be given to the derivation of the relevant equations for the study of cosmic microwave background anisotropies and to their analytical solutions.

2. Introduction

Cosmic Microwave Background (CMB) anisotropies play a special role in cosmology, as they allow an accurate determination of cosmological parameters and may provide a unique probe of the physics of the early universe and in particular of the processes that gave origin to the primordial perturbations.

Cosmological inflation [3] is nowadays considered the dominant paradigm for the generation of the initial seeds for structure formation. In the inflationary picture, the primordial cosmological perturbations are created from quantum fluctuations "redshifted" out of the horizon during an early period of accelerated expansion of the universe, where they remain "frozen". They are observable through CMB temperature anisotropies (and polarization) and the large-scale clustering properties of the matter distribution in the Universe.

This picture has recently received further spectacular confirmations from the results of the Wilkinson Microwave Anisotropy Probe (WMAP) three year set of data [4]. Since the observed cosmological perturbations are of the order of 10^{-5}, one might think that first-order perturbation theory will be adequate for all comparisons with observations. This might not be the case, though. Present [4] and future experiments [5] may be sensitive to the non-linearities of the cosmological perturbations at the level of second- or higher-order perturbation theory. The detection of these non-linearities through the non-Gaussianity (NG) in the CMB [6] has become one of the primary experimental targets.

There is one fundamental reason why a positive detection of NG is so relevant: it might help in discriminating among the various mechanisms for the generation of the cosmological perturbations. Indeed, various models of inflation, firmly rooted in modern particle physics theory, predict a significant amount of primordial NG generated either during or immediately after inflation when the comoving curvature perturbation becomes constant on super-horizon scales [6]. While single-field [7] and two(multi)-field [8] models of inflation predict a tiny level of NG, "curvaton"-type models, in which a significant contribution to the curvature perturbation is generated after the end of slow-roll inflation by the perturbation in a field which has a negligible effect on inflation, may predict a high level of NG [9]. Alternatives to the curvaton model are models where a curvature perturbation mode is generated by an inhomogeneity in the decay rate [10, 11], the mass [12] or the interaction rate [13] of the particles responsible for the reheating after inflation. Other opportunities for generating the curvature perturbations occur at the end of inflation [14], during preheating [15], and at a phase-transition producing cosmic strings [16].

Statistics like the bispectrum and the trispectrum of the CMB can then be used to assess the level of primordial NG on various cosmological scales and to discriminate it from the one induced by secondary anisotropies and systematic effects [6, 17–19]. A positive detection of a primordial NG in the CMB at some level might therefore confirm and/or rule out a whole class of mechanisms by which the cosmological perturbations have been generated.

Despite the importance of NG in CMB anisotropies, little effort has ben made so far to provide accurate theoretical predictions of it. On the contrary, the vast majority of the literature has been devoted to the computation of the bispectrum of either the comovig curvature perturbation or the gravitational potential on large scales within given inflationary models. These, however, are not the physical quantities which are observed. One should instead provide a full prediction for the second-order radiation transfer function. A preliminary step towards this goal has been taken in Ref. [20] (see also [21]) where the full second-order radiation transfer function for the CMB anisotropies on large angular scales in a flat universe filled with matter and cosmological constant was computed, including the second-order generalization of the Sachs-Wolfe effect, both the early and late Integrated Sachs-Wolfe (ISW) effects and the contribution of the second-order tensor modes.

There are many sources of NG in CMB anisotropies, beyond the primordial one. The most relevant sources are the so-called secondary anisotropies, which arise after the last scattering epoch. These anisotropies can be divided into two categories: scattering secondaries, when the CMB photons scatter with electrons along the line of sight, and gravitational secondaries when effects are mediated by gravity [22]. Among the scattering secondaries we may list the thermal Sunyaev-Zeldovich effect, where hot electrons in clusters transfer energy to the CMB photons, the kinetic Sunyaev-Zeldovich effect produced by the bulk motion of the electrons in clusters, the Ostriker-Vishniac effect, produced by bulk motions modulated by linear density perturbations, and effects due to reionization processes. The scattering secondaries are most significant on small angular scales as density inhomogeneities, bulk and thermal motions grow and become sizeable on small length-scales when structure formation proceeds.

Gravitational secondaries arise from the change in energy of photons when the gravitational potential is time-dependent, the ISW effect, and gravitational lensing. At late times, when the Universe becomes dominated by the dark energy, the gravitational potential on linear scales starts to decay, causing the ISW effect mainly on large angular scales. Other secondaries that result from a time dependent potential are the Rees-Sciama effect, produced during the matter-dominated epoch by the time evolution of the potential on non-linear scales.

The fact that the potential never grows appreciably means that most second order effects created by gravitational secondaries are generically small compared to those created by scattering ones. However, when a photon propagates from the last scattering to us, its path may be deflected because of the gravitational lensing. This effect does not create anisotropies, but only modifies existing ones. Since photons with large wavenumbers k are lensed over many regions ($\sim k/H$, where H is the Hubble rate) along the line of sight, the corresponding second-order effect may be sizeable. The three-point function arising from the correlation of the gravitational lensing and ISW effects generated by the matter distribution along the line of sight [23, 24] and the Sunyaev-Zeldovich effect [25] are large and detectable by Planck [26].

Another relevant source of NG comes from the physics operating at the recombination. A naive estimate would tell that these non-linearities are tiny being suppressed by an extra power of the gravitational potential. However, the dynamics at recombination is quite involved because all the non-linearities in the evolution of the baryon-photon fluid at recombination and the ones coming from general relativity should be accounted for. This complicated dynamics might lead to unexpected suppressions or enhancements of the NG at recombination. A step towards the evaluation of the three-point correlation function has been taken in Ref. [27] where some effects were taken into account in the in so-called squeezed triangle limit, corresponding to the case when one wavenumber is much smaller than the other two and was outside the horizon at recombination.

These notes, which are based on Refs. [1, 2], present the computation of the full system of Boltzmann equations, describing the evolution of the photon, baryon and Cold Dark Matter (CDM) fluids, at second order and neglecting polarization, These equations allow to follow the time evolution of the CMB anisotropies at second order on all angular scales from the early epochs, when the cosmological perturbations were generated, to the present time, through the recombination era. These calculations set the stage for the computation of the full second-order radiation transfer function at all scales and for a a generic set of initial conditions specifying the level of primordial non-Gaussianity. Of course on small angular scales, fully non-linear calculations of specific effects like Sunyaev-Zel'dovich, gravitational lensing, etc. would provide a more accurate estimate of the resulting CMB anisotropy, however, as long as the leading contribution to second-order statistics like the bispectrum is concerned, second-order perturbation theory suffices.

These notes are organized as follows. In Section 3 we provide the second-order metric and corresponding Einstein equations. In Section 4 the left-hand-side of the Boltzmann equation for the photon distribution function is derived at second order. The collision term is computed in Section 5. In Section 6 we present the second-order Boltzmann equation for the photon brightness function, its formal solution with the method of the integration along the line of sight and the corresponding hierarchy equations for the multipole moments. Section 7 contains the derivation of the Boltzmann equations at second order for baryons and CDM. Section 8 contains the approximate solution of the Boltzmann equations up to first order. Section 9 contains a brief summary of the results. In Appendix A we give the explicit form of Einstein's equations up to second order, while in Appendix B we provide the first-order solutions of Einstein's equations in various cosmological eras.

In performing the computation presented in these lecture notes, we have mainly followed Ref. [28] (in particular chapter 4), where an excellent derivation of the Boltzmann equations for the baryon-photon fluid at first order is given, and Refs. [1, 2] for their second-order extension. Since the derivation at second order is straightforward, but lenghty, the reader might benefit from reading the appropriate sections of Ref. [28]. In the Conclusions (Section 9) we have also provided a Table which summarizes the many symbols appearing throughout these notes.

3. Perturbing gravity

Before tackling the problem of interest – the computation of the Boltzmann equations for the baryon-photon and CDM fluids – we first provide the necessary tools to deal with perturbed gravity, giving the expressions for the Einstein tensor perturbed up to second order around a flat Friedmann-Robertson-Walker background, and the relevant Einstein equations. In the following we will adopt the Poisson gauge which eliminates one scalar degree of freedom from the g_{0i} component of the metric and one scalar and two vector degrees of freedom from g_{ij}. We will use a metric of the form

$$ds^2 = a^2(\eta)\left[-e^{2\Phi}d\eta^2 + 2\omega_i dx^i d\eta + (e^{-2\Psi}\delta_{ij} + \chi_{ij})dx^i dx^j\right] , \tag{3.1}$$

where $a(\eta)$ is the scale factor as a function of the conformal time η, and ω_i and χ_{ij} are the vector and tensor peturbation modes respectively. Each metric perturbation can be expanded into a linear (first-order) and a second-order part, as for example, the gravitational potential $\Phi = \Phi^{(1)} + \Phi^{(2)}/2$. However in the metric (3.1) the choice of the exponentials greatly helps in computing the relevant expressions, and thus we will always keep them where it is convenient. From Eq. (3.1) one recovers at linear order the well-known longitudinal gauge while at second order, one finds $\Phi^{(2)} = \phi^{(2)} - 2(\phi^{(1)})^2$ and $\Psi^{(2)} = \psi^{(2)} + 2(\psi^{(1)})^2$ where $\phi^{(1)}$, $\psi^{(1)}$ and $\phi^{(2)}$, $\psi^{(2)}$ (with $\phi^{(1)} = \Phi^{(1)}$ and $\psi^{(1)} = \Psi^{(1)}$) are the first and second-order gravitational potentials in the longitudinal (Poisson) gauge adopted in Refs. [6, 29] as far as scalar perturbations are concerned. For the vector and tensor perturbations, we will neglect linear vector modes since they are not produced in standard mechanisms for the generation of cosmological perturbations (as inflation), and we also neglect tensor modes at linear order, since they give a negligible contribution to second order perturbations. Therefore we take ω_i and χ_{ij} to be second-order vector and tensor perturbations of the metric.

Let us now give our definitions for the connection coefficients and their expressions for the metric (3.1). The number of spatial dimensions is $n = 3$. Greek indices ($\alpha, \beta, ..., \mu, \nu, ...$) run from 0 to 3, while latin indices ($a, b, \ldots, i, j, k, \ldots$, $m, n, \ldots$) run from 1 to 3. The space-time metric $g_{\mu\nu}$ has signature $(-, +, +, +)$. The connection coefficients are defined as

$$\Gamma^\alpha_{\beta\gamma} \,=\, \frac{1}{2}\, g^{\alpha\rho}\left(\frac{\partial g_{\rho\gamma}}{\partial x^\beta} \,+\, \frac{\partial g_{\beta\rho}}{\partial x^\gamma} \,-\, \frac{\partial g_{\beta\gamma}}{\partial x^\rho}\right) . \tag{3.2}$$

The Riemann tensor is defined as

$$R^\alpha{}_{\beta\mu\nu} = \Gamma^\alpha_{\beta\nu,\mu} - \Gamma^\alpha_{\beta\mu,\nu} + \Gamma^\alpha_{\lambda\mu}\Gamma^\lambda_{\beta\nu} - \Gamma^\alpha_{\lambda\nu}\Gamma^\lambda_{\beta\mu} . \tag{3.3}$$

The Ricci tensor is a contraction of the Riemann tensor, $R_{\mu\nu} = R^\alpha{}_{\mu\alpha\nu}$ and in terms of the connection coefficient it is given by

$$R_{\mu\nu} \,=\, \partial_\alpha\,\Gamma^\alpha_{\mu\nu} \,-\, \partial_\mu\,\Gamma^\alpha_{\nu\alpha} \,+\, \Gamma^\alpha_{\sigma\alpha}\,\Gamma^\sigma_{\mu\nu} \,-\, \Gamma^\alpha_{\sigma\nu}\,\Gamma^\sigma_{\mu\alpha} . \tag{3.4}$$

The Ricci scalar is the trace of the Ricci tensor, $R = R^{\mu}_{\ \mu}$. The Einstein tensor is defined as $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$.

The Einstein equations are written as $G_{\mu\nu} = \kappa^2 T_{\mu\nu}$, so that $\kappa^2 = 8\pi G_{\rm N}$, where $G_{\rm N}$ is the usual Newtonian gravitational constant.

4. The collisionless Boltzmann equation for photons

We are interested in the anisotropies in the cosmic distribution of photons and inhomogeneities in the matter. Photons are affected by gravity and by Compton scattering with free electrons. The latter are tightly coupled to protons. Both are, of course, affected by gravity. The metric which determines the gravitational forces is influenced by all these components plus CDM (and neutrinos). Our plan is to write down Boltzmann equations for the phase-space distributions of each species in the Universe.

The phase-space distribution of particles $g(x^i, P^\mu, \eta)$ is a function of spatial coordinates x^i, conformal time η, and momentum of the particle $P^\mu = dx^\mu/d\lambda$ where λ parametrizes the particle path. Through the constraint $P^2 \equiv g_{\mu\nu} P^\mu P^\nu = -m^2$, where m is the mass of the particle one can eliminate P^0 and $g(x^i, P^j, \eta)$ gives the number of particles in the differential phase-space volume $dx^1 dx^2 dx^3 dP^1 dP^2 dP^3$. In the following we will denote the distribution function for photons with f.

The photons' distribution evolves according to the Boltzmann equation

$$\frac{df}{d\eta} = \overline{C}[f]\,, \tag{4.1}$$

where the collision term is due to the scattering of photons off free electrons. In the following we will derive the left-hand side of Eq. (4.1) while in the next section we will compute the collision term.

For photons we can impose $P^2 \equiv g_{\mu\nu} P^\mu P^\nu = 0$ and using the metric (3.1) in the conformal time η we find

$$P^2 = a^2 \left[-e^{2\Phi} (P^0)^2 + \frac{p^2}{a^2} + 2\omega_i P^0 P^i \right] = 0\,, \tag{4.2}$$

where we define

$$p^2 = g_{ij} P^i P^j\,. \tag{4.3}$$

From the constraint (4.2)

$$P^0 = e^{-\Phi} \left(\frac{p^2}{a^2} + 2\omega_i P^0 P^i \right)^{1/2}\,. \tag{4.4}$$

Notice that we immediately recover the usual zero and first-order relations $P^0 = p/a$ and $P^0 = p(1 - \Phi^{(1)})/a$.

The components P^i are proportional to pn^i, where n^i is a unit vector with $n^i n_i = \delta_{ij} n^i n^j = 1$. We can write $P^i = Cn^i$, where C is determined by

$$g_{ij} P^i P^j = C^2\, a^2 (e^{-2\Psi} + \chi_{ij} n^i n^j) = p^2\,, \tag{4.5}$$

so that

$$P^i = \frac{p}{a} n^i \left(e^{-2\Psi} + \chi_{km} n^k n^m\right)^{-1/2} = \frac{p}{a} n^i e^{\Psi} \left(1 - \frac{1}{2}\chi_{km} n^k n^m\right)\,, \tag{4.6}$$

where the last equality holds up to second order in the perturbations. Again we recover the zero and first-order relations $P^i = pn^i/a$ and $P^i = pn^i(1 + \Psi^{(1)})/a$ respectively. Thus up to second order we can write

$$P^0 = e^{-\Phi} \frac{p}{a} \left(1 + \omega_i\, n^i\right)\,. \tag{4.7}$$

Eq. (4.6) and (4.7) allow us to replace P^0 and P^i in terms of the variables p and n^i. Therefore, as it is standard in the literature, from now on we will consider the phase-space distribution f as a function of the momentum $\mathbf{p} = pn^i$ with magnitude p and angular direction n^i, $f \equiv f(x^i, p, n^i, \eta)$.

Thus, in terms of these variables, the total time derivative of the distribution function reads

$$\frac{df}{d\eta} = \frac{\partial f}{\partial \eta} + \frac{\partial f}{\partial x^i} \frac{dx^i}{d\eta} + \frac{\partial f}{\partial p} \frac{dp}{d\eta} + \frac{\partial f}{\partial n^i} \frac{dn^i}{d\eta}\,. \tag{4.8}$$

In the following we will compute $dx^i/d\eta$, $dp/d\eta$ and $dn^i/d\eta$.

a) $dx^i/d\eta$:

From

$$P^i = \frac{dx^i}{d\lambda} = \frac{dx^i}{d\eta} \frac{d\eta}{d\lambda} = \frac{dx^i}{d\eta} P^0 \tag{4.9}$$

and from Eq. (4.6) and (4.7)

$$\frac{dx^i}{d\eta} = n^i e^{\Phi + \Psi} \left(1 - \omega_j\, n^j - \frac{1}{2}\chi_{km} n^k n^m\right)\,. \tag{4.10}$$

b) $dp/d\eta$:

For $dp/d\eta$ we make use of the time component of the geodesic equation $dP^0/d\lambda = -\Gamma^0_{\alpha\beta}P^\alpha P^\beta$, where $d/d\lambda = (d\eta/d\lambda)\, d/d\eta = P^0\, d/d\eta$, and

$$\frac{dP^0}{d\eta} = -\Gamma^0_{\alpha\beta}\frac{P^\alpha P^\beta}{P^0}\,, \tag{4.11}$$

Using the metric (3.1) we find

$$\begin{aligned} 2\Gamma^0_{\alpha\beta}P^\alpha P^\beta &= g^{0\nu}\left[2\frac{\partial g_{\nu\alpha}}{\partial x^\beta} - \frac{\partial_{\alpha\beta}}{\partial x^\nu}\right]P^\alpha P^\beta \\ &= 2(\mathcal{H}+\Phi')\left(P^0\right)^2 + 4\Phi_{,i}P^0P^i + 4\mathcal{H}\omega_i P^0 P^i \\ &+ 2e^{-2\Phi}\left[(\mathcal{H}-\Psi')e^{-2\Psi}\delta_{ij} - \omega_{i,j} + \frac{1}{2}\chi'_{ij} + \mathcal{H}\chi_{ij}\right]P^iP^j\,. \end{aligned} \tag{4.12}$$

On the other hand the expression (4.7) of P^0 in terms of p and n^i gives

$$\begin{aligned} \frac{dP^0}{d\eta} &= -\frac{p}{a}\frac{d\Phi}{d\eta}e^{-\Phi}\left(1+\omega_i n^i\right) + e^{-\Phi}\left(1+\omega_i\, n^i\right)\frac{d(p/a)}{d\eta} \\ &+ \frac{p}{a}e^{-\Phi}\frac{d(\omega_i\, n^i)}{d\eta}\,. \end{aligned} \tag{4.13}$$

Thus Eq. (4.11) allows us express $dp/d\eta$ as

$$\frac{1}{p}\frac{dp}{d\eta} = -\mathcal{H} + \Psi' - \Phi_{,i}\, n^i e^{\Phi+\Psi} - \omega'_i\, n^i - \frac{1}{2}\chi'_{ij}n^in^j\,, \tag{4.14}$$

where in Eq. (4.12) we have replaced P^0 and P^i by Eqs. (4.7) and (4.6). Notice that in order to obtain Eq. (4.14) we have used the following expressions for the total time derivatives of the metric perturbations

$$\begin{aligned} \frac{d\Phi}{d\eta} &= \frac{\partial\Phi}{\partial\eta} + \frac{\partial\Phi}{\partial x^i}\frac{dx^i}{d\eta} \\ &= \frac{\partial\Phi}{\partial\eta} + \frac{\partial\Phi}{\partial x^i}n^ie^{\Phi+\Psi}\left(1 - \omega_j\, n^j - \frac{1}{2}\chi_{km}n^kn^m\right) \end{aligned} \tag{4.15}$$

and

$$\frac{d(\omega_i n^i)}{d\eta} = n^i\left(\frac{\partial\omega_i}{\partial\eta} + \frac{\partial\omega_i}{\partial x^j}\frac{dx^j}{d\eta}\right) = \frac{\partial\omega_i}{\partial\eta}n^i + \frac{\partial\omega_i}{\partial x^j}n^in^j\,, \tag{4.16}$$

where we have taken into account that ω_i is already a second-order perturbation so that we can neglect $dn^i/d\eta$ which is at least a first order quantity, and we can take the zero-order expression in Eq. (4.10), $dx^i/d\eta = n^i$. In fact there is also an alternative expression for $dp/d\eta$ which turns out to be useful later and which can be obtained by applying once more Eq. (4.15)

$$\frac{1}{p}\frac{dp}{d\eta} = -\mathcal{H} - \frac{d\Phi}{d\eta} + \Phi' + \Psi' - \omega_i' n^i - \frac{1}{2}\chi_{ij}' n^i n^j \,. \qquad (4.17)$$

c) $dn^i/d\eta$:

We can proceed in a similar way to compute $dn^i/d\eta$. Notice that since in Eq. (4.8) it multiplies $\partial f/\partial n^i$ which is first order, we need only the first order perturbation of $dn^i/d\eta$. We use the spatial components of the geodesic equations $dP^i/d\lambda = -\Gamma^i_{\alpha\beta}P^\alpha P^\beta$ written as

$$\frac{dP^i}{d\eta} = -\Gamma^i_{\alpha\beta}\frac{P^\alpha P^\beta}{P^0} \,. \qquad (4.18)$$

For the right-hand side we find, up to second order,

$$\begin{aligned} 2\Gamma^i_{\alpha\beta}P^\alpha P^\beta &= g^{i\nu}\left[\frac{\partial g_{\alpha\nu}}{\partial x^\beta} + \frac{\partial g_{\beta\nu}}{\partial x^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial x^\nu}\right] P^\alpha P^\beta \qquad (4.19)\\ &= 4(\mathcal{H} - \Psi')P^i P^0 + 2\left(\chi^{i\prime}_{\ k} + \omega^i_{,k} - \omega_k^{\ ,i}\right) P^0 P^k \\ &+ \left(2\frac{\partial\Phi}{\partial x^i}e^{2\Phi+2\Psi} + 2\omega^{i\prime} + 2\mathcal{H}\omega^i\right)\left(P^0\right)^2 - 4\frac{\partial\Psi}{\partial x^k}P^i P^k \\ &+ 2\frac{\partial\Psi}{\partial x^i}\delta_{km}P^k P^m - \left[2\mathcal{H}\omega^i\delta_{jk} - \left(\frac{\partial\chi^i_{\ j}}{\partial x^k} + \frac{\partial\chi^i_{\ k}}{\partial x^j} - \frac{\partial\chi_{jk}}{\partial x_i}\right)\right] P^j P^k \,, \end{aligned}$$

while the expression (4.6) of P^i in terms of our variables p and n^i in the left-hand side of Eq. (4.18) brings

$$\begin{aligned} \frac{dP^i}{d\eta} &= \frac{p}{a}e^\Psi\left[\frac{d\Psi}{d\eta}n^i + \frac{a}{p}\frac{d(p/a)}{d\eta}n^i + \frac{dn^i}{d\eta}\right]\left(1 - \frac{1}{2}\chi_{km}n^k n^m\right) \\ &- \frac{p}{a}n^i e^\Psi \frac{1}{2}\frac{d\left(\chi_{km}n^k n^m\right)}{d\eta} \,. \qquad (4.20) \end{aligned}$$

Thus, using the expression (4.6) for P^i and (4.4) for P^0 in Eq. (4.19), together with the previous result (4.14), the geodesic equation (4.18) gives the following expression $dn^i/d\eta$ (valid up to first order)

$$\frac{dn^i}{d\eta} = \left(\Phi_{,k} + \Psi_{,k}\right) n^k n^i - \Phi^{,i} - \Psi^{,i} \,. \qquad (4.21)$$

To proceed further we now expand the distribution function for photons around the zero-order value $f^{(0)}$ which is that of a Bose-Einstein distribution

$$f^{(0)}(p,\eta) = 2\ \frac{1}{\exp\left\{\frac{p}{T(\eta)}\right\} - 1}\,, \tag{4.22}$$

where $T(\eta)$ is the average (zero-order) temperature and the factor 2 comes from the spin degrees of photons. The perturbed distribution of photons will depend also on x^i and on the propagation direction n^i so as to account for inhomogeneities and anisotropies

$$f(x^i,p,n^i,\eta) = f^{(0)}(p,\eta) + f^{(1)}(x^i,p,n^i,\eta) + \frac{1}{2}f^{(2)}(x^i,p,n^i,\eta)\,, \tag{4.23}$$

where we split the perturbation of the distribution function into a first and a second-order part. The Boltzmann equation up to second order can be written in a straightforward way by recalling that the total time derivative of a given i-th perturbation, as *e.g.* $df^{(i)}/d\eta$ is *at least* a quantity of the i-th order. Thus it is easy to realize, looking at Eq. (4.8), that the left-hand side of Boltzmann equation can be written up to second order as

$$\begin{aligned}
\frac{df}{d\eta} &= \frac{df^{(1)}}{d\eta} + \frac{1}{2}\frac{df^{(2)}}{d\eta} - p\frac{\partial f^{(0)}}{\partial p}\frac{d}{d\eta}\left(\Phi^{(1)} + \frac{1}{2}\Phi^{(2)}\right) \\
&+ p\frac{\partial f^{(0)}}{\partial p}\frac{\partial}{\partial\eta}\left(\Phi^{(1)} + \Psi^{(1)} + \frac{1}{2}\Phi^{(2)} + \frac{1}{2}\Psi^{(2)}\right) \\
&- p\frac{\partial f^{(0)}}{\partial p}\frac{\partial\omega_i}{\partial\eta}n^i - \frac{1}{2}p\frac{\partial f^{(0)}}{\partial p}\frac{\partial\chi_{ij}}{\partial\eta}n^i n^j\,,
\end{aligned} \tag{4.24}$$

where for simplicity in Eq. (4.24) we have already used the background Boltzmann equation $(df/d\eta)|^{(0)} = 0$. In Eq. (4.24) there are all the terms which will give rise to the integrated Sachs-Wolfe effects (corresponding to the terms which explicitly depend on the gravitational perturbations), while other effects, such as the gravitational lensing, are still hidden in the (second-order part) of the first term. In fact in order to obtain Eq. (4.24) we just need for the time being to know the expression for $dp/d\eta$, Eq. (4.17).

5. Collision term

5.1. The Collision Integral

In this section we focus on the collision term due to Compton scattering

$$e(\mathbf{q})\gamma(\mathbf{p}) \longleftrightarrow e(\mathbf{q}')\gamma(\mathbf{p}')\,, \tag{5.1}$$

where we have indicated the momentum of the photons and electrons involved in the collisions. The collision term will be important for small scale anisotropies and spectral distortions. The important point to compute the collision term is that for the epoch of interest very little energy is transferred. Therefore one can proceed by expanding the right hand side of Eq. (4.1) both in the small perturbation, Eq. (4.23), and in the small energy transfer. Part of the computation up to second order has been done in Refs. [30–32] (see also [33]). In particular Refs. [30, 31] are focused on the effects of reionization on the CMB anisotropies thus keeping in the collision term those contributions which are relevant for the small-scale effects due to reionization and neglecting the effects of the metric perturbations on the left-hand side of Eq. (4.1). We will mainly follow the formalism of Ref. [31] and we will keep all the terms arising from the expansion of the collision term up to second order.

The collision term is given (up to second order) by

$$\overline{C}(\mathbf{p}) = C(\mathbf{p}) a e^{\Phi} \,, \tag{5.2}$$

where a is the scale factor and [1]

$$\begin{aligned} C(\mathbf{p}) \quad &= \quad \frac{1}{E(\mathbf{p})} \int \frac{d\mathbf{q}}{(2\pi)^3 2E(\mathbf{q})} \frac{d\mathbf{q}'}{(2\pi)^3 2E(\mathbf{q}')} \frac{d\mathbf{p}'}{(2\pi)^3 2E(\mathbf{p}')} \\ &\times \quad (2\pi)^4 \delta^4(q + p - q' - p') \, |M|^2 \\ &\times \quad \{ g(\mathbf{q}') f(\mathbf{p}') [1 + f(\mathbf{p})] - g(\mathbf{q}) f(\mathbf{p}) [1 + f(\mathbf{p}')] \} \end{aligned} \tag{5.3}$$

where $E(\mathbf{q}) = (q^2 + m_e^2)^{1/2}$, M is the amplitude of the scattering process, $\delta^4(q+p-q'-p') = \delta^3(\mathbf{q}+\mathbf{p}-\mathbf{q}'-\mathbf{p}')\delta(E(\mathbf{q})+p-E(\mathbf{q}')-p')$ ensures the energy-momentum conservation and g is the distribution function for electrons. The Pauli suppression factors $(1 - g)$ have been dropped since for the epoch of interest the density of electrons n_e is low. The electrons are kept in thermal equilibrium by Coulomb interactions with protons and they are non-relativistic, thus we can take a Maxwell-Boltzmann distribution around some bulk velocity $\mathbf{v}$

$$g(\mathbf{q}) = n_e \left(\frac{2\pi}{m_e T_e} \right)^{3/2} \exp \left\{ - \frac{(\mathbf{q} - m_e \mathbf{v})^2}{2 m_e T_e} \right\} \tag{5.4}$$

By using the three dimensional delta function the energy transfer is given by $E(\mathbf{q}) - E(\mathbf{q} + \mathbf{p} - \mathbf{p}')$ and it turns out to be small compared to the typical

[1]The reason why we write the collision term as in Eq. (5.2) is that the starting point of the Boltzmann equation requires differentiation with respect to an affine parameter λ, $df/d\lambda = C'$. In moving to the conformal time η one rewrites the Boltzmann equation as $df/d\eta = C'(P^0)^{-1}$, with $P^0 = d\eta/d\lambda$ given by Eq. (4.7). Taking into account that the collision term is at least of first order, Eq. (5.2) then follows.

thermal energies

$$E(\mathbf{q}) - E(\mathbf{q} + \mathbf{p} - \mathbf{p}') \simeq \frac{(\mathbf{p} - \mathbf{p}') \cdot \mathbf{q}}{m_e} = \mathcal{O}(Tq/m_e)\,. \tag{5.5}$$

In Eq. (5.5) we have used $E(\mathbf{q}) = m_e + q^2/2m_e$ and the fact that, since the scattering is almost elastic ($p \simeq p'$), $(\mathbf{p} - \mathbf{p}')$ is of order $p \sim T$, with q much bigger than $(\mathbf{p} - \mathbf{p}')$. In general, the electron momentum has two contributions, the bulk velocity ($q = m_e v$) and the thermal motion ($q \sim (m_e T)^{1/2}$) and thus the parameter expansion q/m_e includes the small bulk velocity $\mathbf{v}$ and the ratio $(T/m_e)^{1/2}$ which is small because the electrons are non-relativistic.

The expansion of all the quantities entering the collision term in the energy transfer parameter and the integration over the momenta $\mathbf{q}$ and $\mathbf{q}'$ is described in details in Ref. [31]. It is easy to realize that we just need the scattering amplitude up to first order since at zero order $g(\mathbf{q}') = g(\mathbf{q} + \mathbf{p} - \mathbf{p}') = g(\mathbf{q})$ and $\delta(E(\mathbf{q}) + p - E(\mathbf{q}') - p') = \delta(p - p')$ so that all the zero-order quantities multiplying $|M|^2$ vanish. To first order

$$|M|^2 = 6\pi\sigma_T m_e^2[(1 + \cos^2\theta) - 2\cos\theta(1 - \cos\theta)\mathbf{q} \cdot (\hat{\mathbf{p}} + \hat{\mathbf{p}}')/m_e]\,, \tag{5.6}$$

where $\cos\theta = \mathbf{n} \cdot \mathbf{n}'$ is the scattering angle and σ_T the Thompson cross-section. The resulting collision term up to second order is given by [31]

$$\begin{aligned} C(\mathbf{p}) &= \frac{3 n_e \sigma_T}{4p} \int dp' p' \frac{d\Omega'}{4\pi} \Big[c^{(1)}(\mathbf{p}, \mathbf{p}') + c^{(2)}_\Delta(\mathbf{p}, \mathbf{p}') + c^{(2)}_v(\mathbf{p}, \mathbf{p}') \\ &+ c^{(2)}_{\Delta v}(\mathbf{p}, \mathbf{p}') + c^{(2)}_{vv}(\mathbf{p}, \mathbf{p}') + c^{(2)}_K(\mathbf{p}, \mathbf{p}') \Big]\,, \end{aligned} \tag{5.7}$$

where we arrange the different contributions following Ref. [31]. The first order term reads

$$\begin{aligned} c^{(1)}(\mathbf{p}, \mathbf{p}') &= (1 + \cos^2\theta) \Big[\delta(p - p')(f^{(1)}(\mathbf{p}') - f^{(1)}(\mathbf{p})) \\ &+ (f^{(0)}(p') - f^{(0)}(p))(\mathbf{p} - \mathbf{p}') \cdot \mathbf{v} \frac{\partial \delta(p - p')}{\partial p'} \Big]\,, \end{aligned} \tag{5.8}$$

while the second-order terms have been separated into four parts. There is the so-called anisotropy suppression term

$$c^{(2)}_\Delta(\mathbf{p}, \mathbf{p}') = \frac{1}{2}\left(1 + \cos^2\theta\right) \delta(p - p')(f^{(2)}(\mathbf{p}') - f^{(2)}(\mathbf{p}))\,; \tag{5.9}$$

a term which depends on the second-order velocity perturbation defined by the expansion of the bulk flow as $\mathbf{v} = \mathbf{v}^{(1)} + \mathbf{v}^{(2)}/2$

$$c_v^{(2)}(\mathbf{p},\mathbf{p}') = \frac{1}{2}(1+\cos^2\theta)(f^{(0)}(p')-f^{(0)}(p))(\mathbf{p}-\mathbf{p}')\cdot\mathbf{v}^{(2)}\,\frac{\partial\delta(p-p')}{\partial p'}\,; \quad (5.10)$$

a set of terms coupling the photon perturbation to the velocity

$$\begin{aligned} c_{\Delta v}^{(2)}(\mathbf{p},\mathbf{p}') &= \Big(f^{(1)}(\mathbf{p}') - f^{(1)}(\mathbf{p})\Big)\Bigg[\left(1+\cos^2\theta\right)(\mathbf{p}-\mathbf{p}')\cdot\mathbf{v} \\ &\times \frac{\partial\delta(p-p')}{\partial p'} - 2\cos\theta(1-\cos\theta)\delta(p-p')(\mathbf{n}+\mathbf{n}')\cdot\mathbf{v}\Bigg], \end{aligned}$$

and a set of source terms quadratic in the velocity

$$\begin{aligned} c_{vv}^{(2)}(\mathbf{p},\mathbf{p}') &= \Big(f^{(0)}(p') - f^{(0)}(p)\Big)\;(\mathbf{p}-\mathbf{p}')\cdot\mathbf{v}\Bigg[\Big(1+\cos^2\theta\Big) \\ &\times \frac{(\mathbf{p}-\mathbf{p}')\cdot\mathbf{v}}{2}\frac{\partial^2\delta(p-p')}{\partial p'^2} \\ &- 2\cos\theta(1-\cos\theta)(\mathbf{n}+\mathbf{n}')\cdot\mathbf{v}\frac{\partial\delta(p-p')}{\partial p'}\Bigg]. \end{aligned} \quad (5.11)$$

The last contribution are the Kompaneets terms describing spectral distortions to the CMB

$$\begin{aligned} c_K^{(2)}(\mathbf{p},\mathbf{p}') &= \left(1+\cos^2\theta\right)\frac{(\mathbf{p}-\mathbf{p}')^2}{2m_e}\Bigg[\Big(f^{(0)}(p') - f^{(0)}(p)\Big)T_e \qquad (5.12) \\ &\times \frac{\partial^2\delta(p-p')}{\partial p'^2} - \Big(f^{(0)}(p') + f^{(0)}(p) + 2f^{(0)}(p')f^{(0)}(p)\Big) \\ &\times \frac{\partial\delta(p-p')}{\partial p'}\Bigg] + \frac{2(p-p')\cos\theta(1-\cos^2\theta)}{m_e}\Bigg[\delta(p-p') \\ &\times f^{(0)}(p')(1+f^{(0)}(p))\Big(f^{(0)}(p') - f^{(0)}(p)\Big)\frac{\partial\delta(p-p')}{\partial p'}\Bigg]. \end{aligned}$$

Let us make a couple of comments about the various contributions to the collision term. First, notice the term $c_v^{(2)}(\mathbf{p},\mathbf{p}')$ due to second-order perturbations in the velocity of electrons which is absent in Ref. [31]. In standard cosmological scenarios (like inflation) vector perturbations are not generated at linear order, so that linear velocities are irrotational $v^{(1)i} = \partial^i v^{(1)}$. However at second order vector perturbations are generated after horizon crossing as non-linear combinations of primordial scalar modes. Thus we must take into account also a transverse (divergence-free) component, $v^{(2)i} = \partial^i v^{(2)} + v_T^{(2)i}$ with $\partial_i v_T^{(2)i} = 0$. As we will see such vector perturbations will break azimuthal symmetry of the collision term with respect to a given mode $\mathbf{k}$, which instead usually holds at linear order. Secondly, notice that the number density of electrons appearing in Eq. (5.7) must be expanded as $n_e = \bar{n}_e(1+\delta_e)$ and then

$$\delta_e^{(1)}\, c^{(1)}(\mathbf{p},\mathbf{p}') \tag{5.13}$$

gives rise to second-order contributions in addition to the list above, where we split $\delta_e = \delta_e^{(1)} + \delta_e^{(2)}/2$ into a first- and second-order part. In particular the combination with the term proportional to $\mathbf{v}$ in $c^{(1)}(\mathbf{p},\mathbf{p}')$ gives rise to the so-called Vishniac effect, as discussed in Ref. [31].

5.2. Computation of different contributions to the collision term

In the integral (5.7) over the momentum $\mathbf{p}'$ the first-order term gives the usual collision term

$$C^{(1)}(\mathbf{p}) = n_e \sigma_T \left[f_0^{(1)}(p) + \frac{1}{2} f_2^{(1)} P_2(\hat{\mathbf{v}}\cdot\mathbf{n}) - f^{(1)} - p\frac{\partial f^{(0)}}{\partial p}\mathbf{v}\cdot\mathbf{n} \right]\, , \tag{5.14}$$

where one uses the decomposition in Legendre polynomials

$$f^{(1)}(\mathbf{x},p,\mathbf{n}) = \sum_\ell (2\ell+1) f_\ell^{(1)}(p) P_\ell(\cos\vartheta)\, , \tag{5.15}$$

where ϑ is the polar angle of $\mathbf{n}$, $\cos\vartheta = \mathbf{n}\cdot\hat{\mathbf{v}}$.

In the following we compute the second-order collision term separately for the different contributions, using the notation $C(\mathbf{p}) = C^{(1)}(\mathbf{p}) + C^{(2)}(\mathbf{p})/2$. We have not reported the details of the calculation of the first-order term because for its second-order analog, $c_\Delta^{(2)}(\mathbf{p},\mathbf{p}') + c_v^{(2)}(\mathbf{p},\mathbf{p}')$, the procedure is the same. The important difference is that the second-order velocity term includes a vector part, and this leads to a generic angular decomposition of the distribution function (for simplicity drop the time dependence)

$$f^{(i)}(\mathbf{x},p,\mathbf{n}) = \sum_\ell \sum_{m=-\ell}^{\ell} f_{\ell m}^{(i)}(\mathbf{x},p)(-i)^\ell \sqrt{\frac{4\pi}{2\ell+1}} Y_{\ell m}(\mathbf{n})\, , \tag{5.16}$$

such that

$$f^{(i)}_{\ell m} = (-i)^{-\ell}\sqrt{\frac{2\ell+1}{4\pi}}\int d\Omega f^{(i)} Y^*_{\ell m}(\mathbf{n})\,. \tag{5.17}$$

Such a decomposition holds also in Fourier space [34]. The notation at this stage is a bit confusing, so let us restate it: superscripts denote the order of the perturbation; the subscripts refer to the moments of the distribution. Indeed at first order one can drop the dependence on m setting $m=0$ using the fact that the distribution function does not depend on the azimuthal angle ϕ. In this case the relation with $f^{(1)}_l$ is

$$f^{(1)}_{\ell m} = (-i)^{-\ell}(2\ell+1)\delta_{m0}\, f^{(1)}_\ell\,. \tag{5.18}$$

a) $c^{(2)}_\Delta(\mathbf{p},\mathbf{p}')$:
The integral over $\mathbf{p}'$ yields

$$\begin{aligned} C^{(2)}_\Delta(\mathbf{p}) &= \frac{3n_e\sigma_T}{4p}\int dp'p'\frac{d\Omega'}{4\pi}c^{(2)}_\Delta(\mathbf{p},\mathbf{p}') = \frac{3n_e\sigma_T}{4p}\int dp'p'\delta(p-p') \\ &\times \int\frac{d\Omega'}{4\pi}[1+(\mathbf{n}\cdot\mathbf{n}')^2][f^{(2)}(\mathbf{p}')-f^{(2)}(\mathbf{p})]\,. \end{aligned} \tag{5.19}$$

To perform the angular integral we write the angular dependence on the scattering angle $\cos\theta = \mathbf{n}\cdot\mathbf{n}'$ in terms of the Legendre polynomials

$$\begin{aligned} [1+(\mathbf{n}\cdot\mathbf{n}')^2] &= \frac{4}{3}\left[1+\frac{1}{2}P_2(\mathbf{n}\cdot\mathbf{n}')\right] \\ &= \left[1+\frac{1}{2}\sum_{m=-2}^{2}Y_{2m}(\mathbf{n})Y^*_{2m}(\mathbf{n}')\frac{4\pi}{2\ell+1}\right], \end{aligned} \tag{5.20}$$

where in the last step we used the addition theorem for spherical harmonics

$$P_\ell = \frac{4\pi}{2\ell+1}\sum_{m=-2}^{2}Y_{\ell m}(\mathbf{n})Y^*_{\ell m}(\mathbf{n}')\,. \tag{5.21}$$

Using the decomposition (5.17) and the orthonormality of the spherical harmonics we find

$$C^{(2)}_\Delta(\mathbf{p}) = n_e\sigma_T\left[f^{(2)}_{00}(p) - f^{(2)}(\mathbf{p}) - \frac{1}{2}\sum_{m=-2}^{2}\frac{\sqrt{4\pi}}{5^{3/2}}\,f^{(2)}_{2m}(p)\,Y_{2m}(\mathbf{n})\right]. \tag{5.22}$$

It is easy to recover the result for the corresponding first-order contribution in Eq. (5.14) by using Eq. (5.18).
b) $c_v^{(2)}(\mathbf{p},\mathbf{p}')$:

Let us fix for simplicity our coordinates such that the polar angle of $\mathbf{n}'$ is defined by $\mu'=\hat{\mathbf{v}}^{(2)}\cdot\mathbf{n}'$ with ϕ' the corresponding azimuthal angle. The contribution of $c_v^{(2)}(\mathbf{p},\mathbf{p}')$ to the collision term is then

$$\begin{aligned} C_v^{(2)}(\mathbf{p}) &= \frac{3n_e\sigma_T}{4p}v^{(2)}\int dp'p'[f^{(0)}(p')-f^{(0)}(p)]\frac{\partial\delta(p-p')}{\partial p'} \\ &\times \int_{-1}^{1}\frac{d\mu'}{2}(p\mu-p'\mu')\int_0^{2\pi}\frac{d\phi'}{2\pi}[1+(\mathbf{p}\cdot\mathbf{p}')^2]\,. \end{aligned} \tag{5.23}$$

We can use Eq. (5.20) which in our coordinate system reads

$$\frac{4}{3}\left[1+\frac{1}{2}\sum_{m=-2}^{m}\frac{(2-m)!}{(2+m)!}P_2^m(\mathbf{n}\cdot\hat{\mathbf{v}}^{(2)})P_2^m(\mathbf{n}'\cdot\hat{\mathbf{v}}^{(2)})e^{im(\phi'-\phi)}\right]\,, \tag{5.24}$$

so that

$$\int\frac{d\phi'}{2\pi}P_2(\mathbf{n}\cdot\mathbf{n}')=P_2(\mathbf{n}\cdot\hat{\mathbf{v}}^{(2)})P_2(\mathbf{n}'\cdot\hat{\mathbf{v}}^{(2)})=P_2(\mu)P_2(\mu')\,. \tag{5.25}$$

By using the orthonormality of the Legendre polynomials and integrating by parts over p' we find

$$C_v^{(2)}(\mathbf{p})=-n_e\,\sigma_T\,p\frac{\partial f^{(0)}}{\partial p}\mathbf{v}^{(2)}\cdot\mathbf{n}\,. \tag{5.26}$$

As it is clear by the presence of the scalar product $\mathbf{v}^{(2)}\cdot\mathbf{p}$ the final result is independent of the coordinates chosen.
c) $c_{\Delta v}^{(2)}(\mathbf{p},\mathbf{p}')$:

Let us consider the contribution from the first term

$$c_{\Delta v(I)}^{(2)}(\mathbf{p},\mathbf{p}')=\left(1+\cos^2\theta\right)\left(f^{(1)}(\mathbf{p}')-f^{(1)}(\mathbf{p})\right)(\mathbf{p}-\mathbf{p}')\cdot\mathbf{v}\frac{\partial\delta(p-p')}{\partial p'}\,,$$

where the velocity has to be considered at first order. In the integral (5.7) it brings

$$\begin{aligned} \frac{1}{2}C_{\Delta v(I)}^{(2)} &= \frac{3n_e\sigma_T v}{4p}\int dp'p'\frac{\partial\delta(p-p')}{\partial p'}\int_{-1}^{1}\frac{d\mu'}{2}[f^{(1)}(\mathbf{p}')-f^{(1)}(\mathbf{p})] \\ &\times (p\mu-p'\mu')\int_0^{2\pi}\frac{d\phi'}{2\pi}(1+\cos^2\theta)\,, \end{aligned} \tag{5.27}$$

The procedure to do the integral is the same as above. We use the same relations as in Eqs. (5.24) and (5.25) where now the angles are those taken with respect to the first-order velocity. This eliminates the integral over ϕ', and integrating by parts over p' yields

$$\frac{1}{2}C^{(2)}_{\Delta v(I)}(\mathbf{p}) = -\frac{3n_e\sigma_T v}{4p}\int_{-1}^{1}\frac{d\mu'}{2}\left[\frac{4}{3}+\frac{2}{3}P_2(\mu)P_2(\mu')\right] \tag{5.28}$$
$$\times\left[p(\mu-2\mu')(f^{(1)}(p,\mu')-f^{(1)}(p,\mu))+p^2(\mu-\mu')\frac{\partial f^{(1)}(p,\mu')}{\partial p}\right].$$

We now use the decomposition (5.15) and the orthonormality of the Legendre polynomials to find

$$\int\frac{d\mu'}{2}\mu' f^{(1)}(p,\mu')P_2(\mu') = \sum_\ell\int\frac{d\mu'}{2}\mu' P_2(\mu')P_l(\mu')f^{(1)}_\ell(p)$$
$$=\sum_\ell\int\frac{d\mu'}{2}\left[\frac{2}{5}P_1(\mu')+\frac{3}{5}P_3(\mu')\right]P_\ell(\mu')f^{(1)}_\ell(p)$$
$$=\frac{2}{5}f^{(1)}_1(p)+\frac{3}{5}f^{(1)}_3(p)\,, \tag{5.29}$$

where we have used $\mu' P_2(\mu')P_l(\mu') = \frac{2}{5}P_1(\mu')+\frac{3}{5}P_3(\mu')$, with $P_1(\mu')=\mu'$. Thus from Eq. (5.28) we get

$$\begin{aligned}\frac{1}{2}C^{(2)}_{\Delta v(I)}(\mathbf{p}) &= n_e\sigma_T\Bigg\{\mathbf{v}\cdot\mathbf{n}\Bigg[f^{(1)}(\mathbf{p})-f^{(1)}_0(p)-p\frac{\partial f^{(1)}_0(p)}{\partial p}\\ &- \frac{1}{2}P_2(\hat{\mathbf{v}}\cdot\mathbf{n})\left(f^{(1)}_2(p)+p\frac{\partial f^{(1)}_2(p)}{\partial p}\right)\Bigg]\\ &+ v\Bigg[2f^{(1)}_1(p)+p\frac{\partial f^{(1)}_1(p)}{\partial p}+\frac{1}{5}P_2(\hat{\mathbf{v}}\cdot\mathbf{n})\Bigg(2f^{(1)}_1(p)\\ &+ p\frac{\partial f^{(1)}_1(p)}{\partial p}+3f^{(1)}_3(p)+\frac{3}{2}p\frac{\partial f^{(1)}_3(p)}{\partial p}\Bigg)\Bigg]\Bigg\}\,.\end{aligned} \tag{5.30}$$

In $c^{(2)}(\mathbf{p},\mathbf{p}')$ there is a second term

$$c^{(2)}_{\Delta v(II)} = -2\cos\theta(1-\cos\theta)\left(f^{(1)}(\mathbf{p}')-f^{(1)}(\mathbf{p})\right)\delta(p-p')(\mathbf{n}+\mathbf{n}')\cdot\mathbf{v},$$

whose contribution to the collision term is

$$\frac{1}{2}C^{(2)}_{\Delta v(II)}(\mathbf{p}) = -\frac{3n_e\sigma_T v}{2p}\int dp' p'\delta(p-p')\int_{-1}^{1}\frac{d\mu'}{2}(f^{(1)}(\mathbf{p}')$$

$$- \quad f^{(1)}(\mathbf{p}))(\mu+\mu') \int_0^{2\pi} \frac{d\phi'}{2\pi} \cos\theta(1-\cos\theta)\,. \tag{5.31}$$

This integration proceeds through the same steps as for $C^{(2)}_{\Delta v(I)}(\mathbf{p})$. In particular by noting that $\cos\theta(1-\cos\theta) = -1/3 + P_1(\cos\theta) - 2P_3(\cos\theta)/3$, Eqs. (5.24) and (5.25) allows to compute

$$\int \frac{d\phi'}{2\pi} \cos\theta(1-\cos\theta) = -\frac{1}{3} + P_1(\mu)P_1(\mu') - \frac{2}{3}P_2(\mu)P_2(\mu')\,, \tag{5.32}$$

and using the decomposition (5.15) we arrive at

$$\begin{aligned} \frac{1}{2}C^{(2)}_{\Delta v(II)}(\mathbf{p}) &= -n_e\sigma_T \Bigg\{ \mathbf{v}\cdot\mathbf{n}\, f_2^{(1)}(p)(1-P_2(\hat{\mathbf{v}}\cdot\mathbf{n})) \\ &+ \quad v\left[\frac{1}{5}\,P_2(\hat{\mathbf{v}}\cdot\mathbf{n})\,\left(3f_1^{(1)}(p) - 3f_3^{(1)}(p)\right)\right]\Bigg\}\,. \end{aligned} \tag{5.33}$$

We then obtain

$$\begin{aligned} \frac{1}{2}C^{(2)}_{\Delta v}(\mathbf{p}) &= n_e\sigma_T \Bigg\{ \mathbf{v}\cdot\mathbf{n}\Bigg[f^{(1)}(\mathbf{p}) - f_0^{(1)}(p) - p\frac{\partial f_0^{(1)}(p)}{\partial p} - f_2^{(1)}(p) \\ &+ \quad \frac{1}{2}P_2(\hat{\mathbf{v}}\cdot\mathbf{n})\left(f_2^{(1)}(p) - p\frac{\partial f_2^{(1)}(p)}{\partial p}\right)\Bigg] + v\Bigg[2f_1^{(1)}(p) \\ &+ \quad p\frac{\partial f_1^{(1)}(p)}{\partial p} + \frac{1}{5}P_2(\hat{\mathbf{v}}\cdot\mathbf{n})\Bigg(-f_1^{(1)}(p) \\ &+ \quad p\frac{\partial f_1^{(1)}(p)}{\partial p} + 6f_3^{(1)}(p) + \frac{3}{2}p\frac{\partial f_3^{(1)}(p)}{\partial p}\Bigg)\Bigg]\Bigg\}\,. \end{aligned} \tag{5.34}$$

As far as the remaining terms, these have already been computed in Ref. [31] (see also Ref. [30]) and here we just report them

d) $c^{(2)}_{vv}(\mathbf{p},\mathbf{p}')$:

The term proportional to the velocity squared yield a contribution to the collision term

$$\begin{aligned} \frac{1}{2}C^{(2)}_{vv}(\mathbf{p}) &= n_e\sigma_T \Bigg\{ (\mathbf{v}\cdot\mathbf{n})^2 \left[p\frac{\partial f^{(0)}}{\partial p} + \frac{11}{20}p^2\frac{\partial^2 f^{(0)}}{\partial p^2}\right] \\ &+ \quad v^2\left[p\frac{\partial f^{(0)}}{\partial p} + \frac{3}{20}p^2\frac{\partial^2 f^{(0)}}{\partial p^2}\right]\Bigg\}\,. \end{aligned} \tag{5.35}$$

e) $c^{(2)}_{K}(\mathbf{p},\mathbf{p}')$:

The terms responsible for the spectral distortions give

$$\frac{1}{2}C_K^{(2)}(\mathbf{p}) = \frac{1}{m_e^2}\frac{\partial}{\partial p}\left\{p^4\left[T_e\frac{\partial f^{(0)}}{\partial p} + f^{(0)}(1+f^{(0)})\right]\right\} . \tag{5.36}$$

Finally, we write also the part of the collision term coming from Eq. (5.13)

$$\begin{aligned}\delta_e^{(1)}\, c^{(1)}(\mathbf{p},\mathbf{p}') \quad &\rightarrow \quad \delta_e^{(1)}C^{(1)}(\mathbf{p}) = n_e\sigma_T\,\delta_e^{(1)}\Bigg[f_0^{(1)}(p) \\ &+ \quad \frac{1}{2}f_2^{(1)}P_2(\hat{\mathbf{v}}\cdot\mathbf{n}) - f^{(1)} - p\frac{\partial f^{(0)}}{\partial p}\mathbf{v}\cdot\mathbf{n}\Bigg] . \end{aligned} \tag{5.37}$$

5.3. Final expression for the collision term

Summing all the terms we find the final expression for the collision term (5.7) up to second order

$$C(\mathbf{p}) = C^{(1)}(\mathbf{p}) + \frac{1}{2}C^{(2)}(\mathbf{p}) \tag{5.38}$$

with

$$C^{(1)}(\mathbf{p}) = n_e\sigma_T\left[f_0^{(1)}(p) + \frac{1}{2}f_2^{(1)}P_2(\hat{\mathbf{v}}\cdot\mathbf{n}) - f^{(1)} - p\frac{\partial f^{(0)}}{\partial p}\mathbf{v}\cdot\mathbf{n}\right] \tag{5.39}$$

and

$$\begin{aligned}\frac{1}{2}C^{(2)}(\mathbf{p}) \quad &= \quad n_e\sigma_T\Bigg\{\frac{1}{2}f_{00}^{(2)}(p) - \frac{1}{4}\sum_{m=-2}^{2}\frac{\sqrt{4\pi}}{5^{3/2}}\,f_{2m}^{(2)}(p)\,Y_{2m}(\mathbf{n}) \\ &- \quad \frac{1}{2}f^{(2)}(\mathbf{p}) + \delta_e^{(1)}\Bigg[f_0^{(1)}(p) + \frac{1}{2}f_2^{(1)}P_2(\hat{\mathbf{v}}\cdot\mathbf{n}) - f^{(1)} \\ &- \quad p\frac{\partial f^{(0)}}{\partial p}\mathbf{v}\cdot\mathbf{n}\Bigg] - \frac{1}{2}p\frac{\partial f^{(0)}}{\partial p}\mathbf{v}^{(2)}\cdot\mathbf{n} + \mathbf{v}\cdot\mathbf{n}\Bigg[f^{(1)}(\mathbf{p}) \\ &- \quad f_0^{(1)}(p) - p\frac{\partial f_0^{(1)}(p)}{\partial p} - f_2^{(1)}(p) + \frac{1}{2}P_2(\hat{\mathbf{v}}\cdot\mathbf{n}) \\ &\times \quad \left(f_2^{(1)}(p) - p\frac{\partial f_2^{(1)}(p)}{\partial p}\right)\Bigg] + v\Bigg[2f_1^{(1)}(p) + p\frac{\partial f_1^{(1)}(p)}{\partial p} \\ &+ \quad \frac{1}{5}P_2(\hat{\mathbf{v}}\cdot\mathbf{n})\Bigg(-f_1^{(1)}(p) + p\frac{\partial f_1^{(1)}(p)}{\partial p} + 6f_3^{(1)}(p) \\ &+ \quad \frac{3}{2}p\frac{\partial f_3^{(1)}(p)}{\partial p}\Bigg)\Bigg] + (\mathbf{v}\cdot\mathbf{n})^2\left[p\frac{\partial f^{(0)}}{\partial p} + \frac{11}{20}p^2\frac{\partial^2 f^{(0)}}{\partial p^2}\right]\end{aligned}$$

$$
\begin{aligned}
&+ \quad v^2\left[p\frac{\partial f^{(0)}}{\partial p}+\frac{3}{20}p^2\frac{\partial^2 f^{(0)}}{\partial p^2}\right] \\
&+ \quad \frac{1}{m_e^2}\frac{\partial}{\partial p}\left[p^4\left(T_e\frac{\partial f^{(0)}}{\partial p}+f^{(0)}(1+f^{(0)})\right)\right]\Bigg\}. \qquad (5.40)
\end{aligned}
$$

Notice that there is an internal hierarchy, with terms which do not depend on the baryon velocity $\mathbf{v}$, terms proportional to $\mathbf{v}\cdot\mathbf{n}$ and then to $(\mathbf{v}\cdot\mathbf{n})^2$, v and v^2 (apart from the Kompaneets terms). In particular notice the term proportional to $\delta_e^{(1)}\mathbf{v}\cdot\mathbf{n}$ is the one corresponding to the Vishniac effect. We point out that we have kept all the terms up to second order in the collision term. In Refs. [30, 31] many terms coming from $c_{\Delta v}^{(2)}$ have been dropped mainly because these terms are proportional to the photon distribution function $f^{(1)}$ which on very small scales (those of interest for reionization) is suppressed by the diffusion damping. Here we want to be completely general and we have to keep them.

6. The Brightness equation

6.1. *First order*

The Boltzmann equation for photons is obtained by combining Eq. (4.24) with Eqs. (5.39)-(5.40). At first order the left-hand side reads

$$
\frac{df}{d\eta} = \frac{df^{(1)}}{d\eta} - p\frac{\partial f^{(0)}}{\partial p}\frac{\partial \Phi^{(1)}}{\partial x^i}\frac{dx^i}{d\eta} + p\frac{\partial f^{(0)}}{\partial p}\frac{\partial \Psi^{(1)}}{\partial \eta}\,. \qquad (6.1)
$$

At first order it is useful to characterize the perturbations to the Bose-Einstein distribution function (4.22) in terms of a perturbation to the temperature as

$$
f(x^i,p,n^i,\eta) = 2\left[\exp\left\{\frac{p}{T(\eta)(1+\Theta^{(1)})}\right\}-1\right]^{-1}. \qquad (6.2)
$$

Thus it turns out that

$$
f^{(1)} = -p\frac{\partial f^{(0)}}{\partial p}\Theta^{(1)}\,, \qquad (6.3)
$$

where we have used the fact that $\partial f/\partial\Theta|_{\Theta=0} = -p\partial f^{(0)}/\partial p$. In terms of this variable $\Theta^{(1)}$ the linear collision term (5.39) will now become proportional to $-p\partial f^{(0)}/\partial p$ which contains the only explicit dependence on p, and the same happens for the left-hand side, Eq. (6.1). This is telling us that at first order $\Theta^{(1)}$ does not depend on p but only on x^i, n^i, η, $\Theta^{(1)} = \Theta^{(1)}(x^i, n^i, \tau)$. This is well known and the physical reason is that at linear order there is no energy transfer in Compton collisions between photons and electrons. Therefore, the Boltzmann equation for $\Theta^{(1)}$ reads

$$\frac{\partial\Theta^{(1)}}{\partial\eta}+n^i\frac{\partial\Theta^{(1)}}{\partial x^i}+\frac{\partial\Phi^{(1)}}{\partial x^i}n^i-\frac{\partial\Psi^{(1)}}{\partial\eta}$$
$$=n_e\sigma_T a\left[\Theta_0^{(1)}+\frac{1}{2}\Theta_2^{(1)}P_2(\hat{\mathbf{v}}\cdot\mathbf{n})-\Theta^{(1)}+\mathbf{v}\cdot\mathbf{n}\right], \tag{6.4}$$

where we made us of $f_\ell^{(1)}=-p\partial f^{(0)}/\partial p\Theta_\ell^{(1)}$, according to the decomposition of Eq. (5.15), and we have taken the zero-order expressions for $dx^i/d\eta$, dropping the contribution from $dn^i/d\eta$ in Eq. (4.8) since it is already first-order.

Notice that, since $\Theta^{(1)}$ is independent of p, it is equivalent to consider the quantity

$$\Delta^{(1)}(x^i,n^i,\tau)=\frac{\int dpp^3f^{(1)}}{\int dpp^3f^{(0)}}, \tag{6.5}$$

being $\Delta^{(1)}=4\Theta^{(1)}$ at this order. The physical meaning of $\Delta^{(1)}$ is that of a fractional energy perturbation (in a given direction). From Eq. (4.24) another way to write an equation for $\Delta^{(1)}$ – the so-called brightness equation – is

$$\frac{d}{d\eta}\left[\Delta^{(1)}+4\Phi^{(1)}\right]-4\frac{\partial}{\partial\eta}\left(\Phi^{(1)}+\Psi^{(1)}\right)$$
$$=n_e\sigma_T a\left[\Delta_0^{(1)}+\frac{1}{2}\Delta_2^{(1)}P_2(\hat{\mathbf{v}}\cdot\mathbf{n})-\Delta^{(1)}+4\mathbf{v}\cdot\mathbf{n}\right]. \tag{6.6}$$

6.2. Second order

The previous results show that at linear order the photon distribution function has a Planck spectrum with the temperature that at any point depends on the photon direction. At second order one could characterize the perturbed photon distribution function in a similar way as in Eq. (6.2)

$$f(x^i,p,n^i,\eta)=2\left[\exp\left\{\frac{p}{T(\eta)\,e^{\Theta}}-1\right\}\right]^{-1}, \tag{6.7}$$

where by expanding $\Theta=\Theta^{(1)}+\Theta^{(2)}/2+\ldots$ as usual one recovers the first-order expression. For example, in terms of Θ, the perturbation of $f^{(1)}$ is given by Eq. (6.3), while at second order

$$\frac{f^{(2)}}{2}=-\frac{p}{2}\frac{\partial f^{(0)}}{\partial p}\Theta^{(2)}+\frac{1}{2}\left(p^2\frac{\partial^2 f^{(0)}}{\partial p^2}+p\frac{\partial f^{(0)}}{\partial p}\right)\left(\Theta^{(1)}\right)^2. \tag{6.8}$$

However, as discussed in details in Refs. [30, 31], now the second-order perturbation $\Theta^{(2)}$ will not be momentum independent because the collision term in the equation for $\Theta^{(2)}$ does depend explicitly on p (defining the combination $-(p\partial f^{(0)}/\partial p)^{-1} f^{(2)}$ does not lead to a second-order momentum independent equation as above). Such dependence is evident, for example, in the terms of $C^{(2)}(\mathbf{p})$, Eq. (5.40), proportional to v or v^2, and in the Kompaneets terms. The physical reason is that at the non-linear level photons and electrons do exchange energy during Compton collisions. As a consequence spectral distorsions are generated. For example, in the isotropic limit, only the Kompaneets terms survive giving rise to the Sunyaev-Zeldovich distorsions. As discussed in Ref. [30], the Sunyaev-Zeldovich distorsions can also be obtained with the correct coefficients by replacing the average over the direction electron $\langle v^2 \rangle$ with the mean squared thermal velocity $\langle v_{th}^2 \rangle = 3T_e/m_e$ in Eq. (5.40). This is due simply to the fact that the distinction between thermal and bulk velocity of the electrons is just for convenience. This fact also shows that spectral distorsions due to the bulk flow (kinetic Sunyaev-Zeldovich) has the same form as the thermal effect. Thus spectral distorsions can be in general described by a global Compton y-parameter (see Ref. [30] for a full discussion of spectral distorsions). However in the following we will not be interested in the frequency dependence but only in the anisotropies of the radiation distribution. Therefore we can integrate over the momentum p and define [30, 31]

$$\Delta^{(2)}(x^i, n^i, \tau) = \frac{\int dp p^3 f^{(2)}}{\int dp p^3 f^{(0)}}\,, \tag{6.9}$$

as in Eq. (6.5).

Integration over p of Eqs. (4.24)-(5.40) is straightforward using the following relations

$$\begin{aligned} \int dp p^3 p \frac{\partial f^{(0)}}{\partial p} &= -4\,N\,; \qquad \int dp p^3 p^2 \frac{\partial^2 f^{(0)}}{\partial p^2} = 20\,N\,; \\ \int dp p^3 f^{(1)} &= N\Delta^{(1)}\,; \qquad \int dp p^3 p \frac{\partial f^{(1)}}{\partial p} = -4\,N\Delta^{(1)}\,. \end{aligned} \tag{6.10}$$

Here $N = \int dp p^3 f^{(0)}$ is the normalization factor (it is just proportional the background energy density of photons $\bar{\rho}_\gamma$). At first order one recovers Eq. (6.6). At second order we find

$$\begin{aligned} &\frac{1}{2}\frac{d}{d\eta}\left[\Delta^{(2)} + 4\Phi^{(2)}\right] + \frac{d}{d\eta}\left[\Delta^{(1)} + 4\Phi^{(1)}\right] - 4\Delta^{(1)}\left(\Psi^{(1)\prime} - \Phi^{(1)}_{,i} n^i\right) \\ &-2\frac{\partial}{\partial\eta}\left(\Psi^{(2)} + \Phi^{(2)}\right) + 4\frac{\partial\omega_i}{\partial\eta} n^i + 2\frac{\partial\chi_{ij}}{\partial\eta} n^i n^j = \end{aligned}$$

$$= -\frac{\tau'}{2}\left[\Delta^{(2)}_{00} - \Delta^{(2)} - \frac{1}{2}\sum_{m=-2}^{2}\frac{\sqrt{4\pi}}{5^{3/2}}\,\Delta^{(2)}_{2m}\,Y_{2m}(\mathbf{n}) + 2(\delta^{(1)}_e + \Phi^{(1)})\right.$$
$$\left(\Delta^{(1)}_0 + \frac{1}{2}\Delta^{(1)}_2 P_2(\hat{\mathbf{v}}\cdot\mathbf{n}) - \Delta^{(1)} + 4\mathbf{v}\cdot\mathbf{n}\right) + 4\mathbf{v}^{(2)}\cdot\mathbf{n}$$
$$+2(\mathbf{v}\cdot\mathbf{n})\left[\Delta^{(1)} + 3\Delta^{(1)}_0 - \Delta^{(1)}_2\left(1 - \frac{5}{2}P_2(\hat{\mathbf{v}}\cdot\mathbf{n})\right)\right]$$
$$\left.-v\Delta^{(1)}_1\left(4 + 2P_2(\hat{\mathbf{v}}\cdot\mathbf{n})\right) + 14(\mathbf{v}\cdot\mathbf{n})^2 - 2v^2\right], \tag{6.11}$$

where we have expanded the angular dependence of Δ as in Eq. (5.16)

$$\Delta^{(i)}(\mathbf{x},\mathbf{n}) = \sum_\ell \sum_{m=-\ell}^{\ell} \Delta^{(i)}_{\ell m}(\mathbf{x})(-i)^\ell \sqrt{\frac{4\pi}{2\ell+1}} Y_{\ell m}(\mathbf{n})\,, \tag{6.12}$$

with

$$\Delta^{(i)}_{\ell m} = (-i)^{-\ell}\sqrt{\frac{2\ell+1}{4\pi}}\int d\Omega \Delta^{(i)} Y^*_{\ell m}(\mathbf{n})\,, \tag{6.13}$$

where we recall that the superscript stands by the order of the perturbation. At first order one can drop the dependence on m setting $m=0$ so that $\Delta^{(1)}_{\ell m} = (-i)^{-\ell}(2\ell+1)\delta_{m0}\,\Delta^{(1)}_\ell$. In Eq. (6.11) we have introduced the differential optical depth

$$\tau' = -\bar{n}_e \sigma_T a\,. \tag{6.14}$$

It is understood that on the left-hand side of Eq. (6.11) one has to pick up for the total time derivatives only those terms which contribute to second order. Thus we have to take

$$\frac{1}{2}\frac{d}{d\eta}\left[\Delta^{(2)} + 4\Phi^{(2)}\right] + \frac{d}{d\eta}\left[\Delta^{(1)} + 4\Phi^{(1)}\right]\Big|^{(2)} \tag{6.15}$$
$$= \frac{1}{2}\left(\frac{\partial}{\partial\eta} + n^i\frac{\partial}{\partial x^i}\right)\left(\Delta^{(2)} + 4\Phi^{(2)}\right) + n^i(\Phi^{(1)} + \Psi^{(1)})$$
$$\times\partial_i(\Delta^{(1)} + 4\Phi^{(1)}) + \left[(\Phi^{(1)}_{,j} + \Psi^{(1)}_{,j})n^i n^j - (\Phi^{,i} + \Psi^{,i})\right]\frac{\partial\Delta^{(1)}}{\partial n^i}\,,$$

where we used Eqs. (4.10) and (4.21). Notice that we can write $\partial\Delta^{(1)}/\partial n^i = (\partial\Delta^{(1)}/\partial x^i)(\partial x^i/\partial n^i) = (\partial\Delta^{(1)}/\partial x^i)(\eta - \eta_i)$, from the integration in time of Eq. (4.10) at zero-order when n^i is constant in time.

6.3. Hierarchy equations for multipole moments

Let us now move to Fourier space. In the following, for a given wave-vector $\mathbf{k}$ we will choose the coordinate system such that $\mathbf{e}_3 = \hat{\mathbf{k}}$ and the polar angle of the photon momentum is ϑ, with $\mu = \cos\vartheta = \hat{\mathbf{k}} \cdot \mathbf{n}$. Then Eq. (6.11) can be written as

$$\Delta^{(2)\prime} + ik\mu\Delta^{(2)} - \tau'\Delta^{(2)} = S(\mathbf{k}, \mathbf{n}, \eta)\,, \tag{6.16}$$

where $S(\mathbf{k}, \mathbf{n}, \eta)$ can be easily read off Eq. (6.11). We now expand the temperature anisotropy in the multipole moments $\Delta^{(2)}_{\ell m}$ in order to obtain a system of coupled differential equations. By applying the angular integral of Eq. (6.13) to Eq. (6.16) we find

$$\Delta^{(2)\prime}_{\ell m}(\mathbf{k}, \eta) = k\left[\frac{\kappa_{\ell m}}{2\ell - 1}\Delta^{(2)}_{\ell-1,m} - \frac{\kappa_{\ell+1,m}}{2\ell+3}\Delta^{(2)}_{\ell+1,m}\right] + \tau'\Delta^{(2)}_{\ell m} + S_{\ell m} \tag{6.17}$$

where the expansion coefficients of the source term are given by

$$\begin{aligned}
S_{\ell m} &= \left(4\Psi^{(2)\prime} - \tau'\Delta^{(2)}_{00}\right)\delta_{\ell 0}\delta_{m0} + 4k\Phi^{(2)}\delta_{\ell 1}\delta_{m0} - 4\,\omega'_{\pm 1}\delta_{\ell 1} \\
&- 8\tau' v^{(2)}_m \delta_{\ell 1} - \frac{\tau'}{10}\Delta^{(2)}_{\ell m}\,\delta_{\ell 2} - 2\chi'_{\pm 2}\,\delta_{\ell 2} \\
&- 2\tau' \int \frac{d^3k_1}{(2\pi)^3}\left[v^{(1)}_0(\mathbf{k}_1)v^{(1)}_0(\mathbf{k}_2)\hat{\mathbf{k}}_2\cdot\hat{\mathbf{k}}_1 + (\delta^{(1)}_e(\mathbf{k}_1) + \Phi^{(1)}(\mathbf{k}_1))\right. \\
&\times \left.\Delta^{(1)}_0(\mathbf{k}_2) - i\frac{2}{3}v(\mathbf{k}_1)\Delta^{(1)}_{10}(\mathbf{k}_2)\right]\delta_{\ell 0}\delta_{m0} \\
&+ 16k\int\frac{d^3k_1}{(2\pi)^3}\left[\Phi^{(1)}(\mathbf{k}_1)\Phi^{(1)}(\mathbf{k}_2)\right]\delta_{\ell 1}\delta_{m0} - 2\left[(\Psi^{(1)}\nabla\Phi^{(1)})_m\right. \\
&+ \left. 8\tau'[(\delta^{(1)}_e + \Phi)\mathbf{v}]_m + 6\tau'(\Delta^{(1)}_0\mathbf{v})_m - 2\tau'(\Delta^{(1)}_2\mathbf{v})_m\right]\delta_{\ell 1} + \tau' \\
&\times \int\frac{d^3k_1}{(2\pi)^3}\left[(\delta^{(1)}_e(\mathbf{k}_1) + \Phi^{(1)}(\mathbf{k}_1))\Delta^{(1)}_2(\mathbf{k}_2) - i\frac{2}{3}v(\mathbf{k}_1)\Delta^{(1)}_{10}(\mathbf{k}_2)\right] \\
&\times \delta_{\ell 2}\,\delta_{m0} + \int\frac{d^3k_1}{(2\pi)^3}\left[8\Psi^{(1)}(\mathbf{k}_1) + 2\tau'(\delta^{(1)}_e(\mathbf{k}_1) + \Phi^{(1)}(\mathbf{k}_1))\right. \\
&- \left.(\eta - \eta_i)(\Psi^{(1)} + \Phi^{(1)})(\mathbf{k}_1)\,\mathbf{k}_1\cdot\mathbf{k}_2\right]\Delta^{(1)}_{\ell 0}(\mathbf{k}_2)\delta_{m0} - i(-i)^{-\ell} \\
&\times (-1)^{-m}(2\ell+1)\sum_{\ell''}\sum_{m'=-1}^{1}(2\ell''+1)\left[8\Delta^{(1)}_{\ell''}\nabla\Phi^{(1)} - 2(\Phi^{(1)}\right.
\end{aligned}$$

$$
\begin{aligned}
+ & \quad \Psi^{(1)})\nabla\Delta^{(1)}_{\ell''}\Big]_{m'} \begin{pmatrix} \ell'' & 1 & \ell \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \ell'' & 1 & \ell \\ 0 & m' & -m \end{pmatrix} \\
+ & \quad \tau' i(-i)^{-\ell}(-1)^{-m}(2\ell+1)\sum_{\ell''}\sum_{m'=-1}^{1}(2\ell''+1)\Big[2\Delta^{(1)}_{\ell''}\mathbf{v} \\
+ & \quad 5\delta_{\ell''2}\,\Delta^{(1)}_{2}\mathbf{v}\Big]_{m'} \begin{pmatrix} \ell'' & 1 & \ell \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \ell'' & 1 & \ell \\ 0 & m' & -m \end{pmatrix} \\
+ & \quad 14\tau'(-i)^{-\ell}(-1)^{-m}\sum_{m',m''=-1}^{1}\int\frac{d^3k_1}{(2\pi)^3}\Big[v^{(1)}_0(\mathbf{k}_1)\frac{v^{(1)}_0(\mathbf{k}_2)}{k_2} \\
\times & \quad \frac{4\pi}{3}Y^*_{1m'}(\hat{\mathbf{k}}_1)\left(kY^*_{1m''}(\hat{\mathbf{k}})-k_1Y^*_{1m''}(\hat{\mathbf{k}}_1)\right)\Big] \\
\times & \quad \begin{pmatrix} 1 & 1 & \ell \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & \ell \\ m' & m'' & -m \end{pmatrix} + 2(\eta-\eta_i)(-i)^{-\ell} \\
\times & \quad (-1)^{-m}\sqrt{\frac{2\ell+1}{4\pi}}\sum_L\sum_{m',m''=-1}^{1}\int\frac{d^3k_1}{(2\pi)^3}\sqrt{\frac{4\pi}{2L+1}}\left(\frac{4\pi}{3}\right)^2 \\
\times & \quad \Delta^{(1)}_L(\mathbf{k}_1)(\Phi^{(1)}+\Psi^{(1)})(\mathbf{k}_2)k_1Y^*_{1m'}(\hat{\mathbf{k}}_1)\left(kY^*_{1m''}(\hat{\mathbf{k}})\right. \\
- & \quad \left. k_1Y^*_{1m''}(\hat{\mathbf{k}}_1)\right)\int d\Omega Y_{1m'}(\mathbf{n})Y_{1m''}(\mathbf{n})Y_{L0}(\mathbf{n})Y_{\ell-m}(\mathbf{n})\,, \qquad (6.18)
\end{aligned}
$$

where $\mathbf{k}_2=\mathbf{k}-\mathbf{k}_1$ and $k_2=|\mathbf{k}_2|$. In Eq. (6.18) it is understood that $|m|\leq\ell$.

Let us explain the notations we have adopted in writing Eq. (6.18). The baryon velocity at linear order is irrotational, meaning that it is the gradient of a potential, and thus in Fourier space it is parallel to $\hat{\mathbf{k}}$, and following the same notation of Ref. [37], we write

$$
\mathbf{v}^{(1)}(\mathbf{k}) = -iv^{(1)}_0(\mathbf{k})\hat{\mathbf{k}}\,. \qquad (6.19)
$$

The second-order velocity perturbation will contain a transverse (divergence-free) part whose components are orthogonal to $\hat{\mathbf{k}}=\mathbf{e}_3$, and we can write

$$
\mathbf{v}^{(2)}(\mathbf{k}) = -iv^{(2)}_0(\mathbf{k})\mathbf{e}_3 + \sum_{m=\pm1} v^{(2)}_m\,\frac{\mathbf{e}_2\mp i\mathbf{e}_1}{\sqrt{2}}\,, \qquad (6.20)
$$

where $\mathbf{e}_i$ form an orhtonormal basis with $\hat{\mathbf{k}}$. The second-order perturbation ω_i is decomposed in a similar way, with $\omega_{\pm1}$ the corresponding components (in this case in the Poisson gauge there is no scalar component). Similarly for the tensor perturbation χ_{ij} we have indicated its amplitudes as $\chi_{\pm 2}$ in the decomposition [36]

$$\chi_{ij} = \sum_{m=\pm 2} -\sqrt{\frac{3}{8}}\,\chi_m(\mathbf{e}_1 \pm i\mathbf{e}_2)_i(\mathbf{e}_1 \pm i\mathbf{e}_2)_j\,. \tag{6.21}$$

We have taken into account that in the gravitational part of the Boltzmann equation and in the collsion term there are some terms, like $\delta_e^{(1)}\mathbf{v}$, which still can be decomposed in the scalar and transverse parts in Fourier space as in Eq. (6.20). For a generic quantity $f(\mathbf{x})\mathbf{v}$ we have indicated the corresponding scalar and vortical components with $(f\mathbf{v})_m$ and their explicit expression is easily found by projecting the Fourier modes of $f(\mathbf{x})\mathbf{v}$ along the $\hat{\mathbf{k}} = \mathbf{e}_3$ and $(\mathbf{e}_2 \mp i\mathbf{e}_1)$ directions

$$(f\mathbf{v})_m(\mathbf{k}) = \int \frac{d^3k_1}{(2\pi)^3} v_0^{(1)}(\mathbf{k}_1) f(\mathbf{k}_2) Y^*_{1m}(\hat{\mathbf{k}}_1)\sqrt{\frac{4\pi}{3}}\,. \tag{6.22}$$

Similarly for a term like $f(\mathbf{x})\nabla g(\mathbf{x})$ we used the notation

$$(f\nabla g)_m(\mathbf{k}) = -\int \frac{d^3k_1}{(2\pi)^3} k_1 g(\mathbf{k}_1) f(\mathbf{k}_2) Y^*_{1m}(\hat{\mathbf{k}}_1)\sqrt{\frac{4\pi}{3}}\,. \tag{6.23}$$

Finally, the first term on the right-hand side of Eq. (6.17) has been obtained by using the relation

$$\begin{aligned} i\mathbf{k}\cdot\mathbf{n}\,\Delta^{(2)}(\mathbf{k}) &= \sum_{\ell m} \Delta^{(2)}_{\ell m}(\mathbf{k}) \frac{k}{2\ell+1}\left[\kappa_{\ell m}\tilde{G}_{\ell-1,m} - \kappa_{\ell+1,m}\tilde{G}_{\ell+1,m}\right] \\ &= k\sum_{\ell m}\left[\frac{\kappa_{\ell m}}{2\ell-1}\Delta^{(2)}_{\ell-1,m} - \frac{\kappa_{\ell m}}{2\ell+3}\Delta^{(2)}_{\ell+1,m}\right]\tilde{G}_{\ell m}\,, \end{aligned} \tag{6.24}$$

where $\tilde{G}_{\ell m} = (-i)^\ell\sqrt{4\pi/(2\ell+1)}Y_{\ell m}(\mathbf{n})$ is the angular mode for the decomposition (6.12) and $\kappa_{\ell m} = \sqrt{l^2 - m^2}$. This relation has been discussed in Refs. [34, 36] and corresponds to the term $n^i\partial\Delta^{(2)}/\partial x^i$ in Eq. (6.11).

As expected, at second order we recover some intrinsic effects which are characteristic of the linear regime. In Eq. (6.17) the relation (6.24) represents the free streaming effect: when the radiation undergoes free-streaming, the inhomogeneities of the photon distribution are seen by the obsever as angular anisotropies. At first order it is responsible for the hierarchy of Boltzmann equations coupling the different ℓ modes, and it represents a projection effect of fluctuations on a scale k onto the angular scale $\ell \sim k\eta$. The term $\tau'\Delta^{(2)}_{\ell m}$ causes an exponential suppression of anisotropies in the absence of the source term $S_{\ell m}$. The first line of the source term (6.18) just reproduces the expression of the first order case. Of course the dynamics of the second-order metric and baryon-velocity perturbations which appear will be different and governed by the second-order Einstein equations and continuity equations. The remaining terms in the source are second-order effects generated as non-linear combinations of the primordial (first-order) perturbations. We have ordered them according to the increasing number of ℓ modes they contribute to. Notice in particular that they involve the first-order anisotropies $\Delta_\ell^{(1)}$ and as a consequence such terms contribute to generate the hierarchy of equations (apart from the free-streaming effect). The source term contains additional scattering processes and gravitational effects. On large scales (above the horizon at recombination) we can say that the main effects are due to gravity, and they include the Sachs-Wolfe and the (late and early) Sachs-Wolfe effect due to the redshift photons suffer when travelling through the second-order gravitational potentials. These, toghether with the contribution due to the second-order tensor modes, have been already studied in details in Ref. [20]. Another important gravitational effect is that of lensing of photons as they travel from the last scattering surface to us. A contribution of this type is given by the last term of Eq. (6.18).

6.4. Integral solution of the second-order Boltzmann equation

As in linear theory, one can derive an integral solution of the Boltzmann equation (6.11) in terms of the source term S. Following the standard procedure (see *e.g.* Ref. [28, 35]) for linear perturbations, we write the left-hand side as $\Delta^{(2)\prime} + ik\mu\Delta^{(2)} - \tau'\Delta^{(2)} = e^{-ik\mu\eta+\tau} d[\Delta^{(2)} e^{ik\mu\eta-\tau}]/d\eta$ in order to derive the integral solution

$$\Delta^{(2)}(\mathbf{k}, \mathbf{n}, \eta_0) = \int_0^{\eta_0} d\eta S(\mathbf{k}, \mathbf{n}, \eta) e^{ik\mu(\eta-\eta_0)-\tau} , \tag{6.25}$$

where η_0 stands by the present time. The expression of the photon moments $\Delta_{\ell m}^{(2)}$ can be obtained as usual from Eq. (6.13). In the previous section we have already found the coefficients for the decomposition of source term S

$$S(\mathbf{k}, \mathbf{n}, \eta) = \sum_\ell \sum_{m=-\ell}^{\ell} S_{\ell m}(\mathbf{k}, \eta)(-i)^\ell \sqrt{\frac{4\pi}{2\ell+1}} Y_{\ell m}(\mathbf{n}) . \tag{6.26}$$

In Eq. (6.25) there is an additional angular dependence in the exponential. It is easy to take it into account by recalling that

$$e^{i\mathbf{k}\cdot\mathbf{x}} = \sum_\ell (i)^\ell (2\ell+1) j_\ell(kx) P_\ell(\hat{\mathbf{k}} \cdot \hat{\mathbf{x}}) . \tag{6.27}$$

Thus the angular integral (6.13) is computed by using the decomposition of the source term (6.26) and Eq. (6.27)

$$\begin{aligned}\Delta^{(2)}_{\ell m}(\mathbf{k},\eta_0) &= (-1)^{-m}(-i)^{-\ell}(2\ell+1)\int_0^{\eta_0} d\eta\, e^{-\tau(\eta)} \\ &\times \sum_{\ell_2}\sum_{m_2=-\ell_2}^{\ell_2}(-i)^{\ell_2}S_{\ell_2 m_2}\sum_{\ell_1} i^{\ell_1} j_{\ell_1}(k(\eta-\eta_0)) \\ &\times (2\ell_1+1)\begin{pmatrix}\ell_1 & \ell_2 & \ell\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}\ell_1 & \ell_2 & \ell\\ 0 & m_2 & -m\end{pmatrix}, \qquad (6.28)\end{aligned}$$

where the Wigner $3-j$ symbols appear because of the Gaunt integrals

$$\begin{aligned}\mathcal{G}^{m_1 m_2 m_3}_{l_1 l_2 l_3} &\equiv \int d^2\hat{\mathbf{n}} Y_{l_1 m_1}(\hat{\mathbf{n}})Y_{l_2 m_2}(\hat{\mathbf{n}})Y_{l_3 m_3}(\hat{\mathbf{n}}) \\ &= \sqrt{\frac{(2l_1+1)\,(2l_2+1)\,(2l_3+1)}{4\pi}} \\ &\times \begin{pmatrix}\ell_1 & \ell_2 & \ell_3\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}\ell_1 & \ell_2 & \ell_3\\ m_1 & m_2 & m_3\end{pmatrix}, \qquad (6.29)\end{aligned}$$

Since the second of the Wigner 3-j symbols in Eq. (6.28) is nonzero only if $m=m_2$, our solution can be rewritten to recover the corresponding expression found for linear anisotropies in Refs. [34, 36]

$$\frac{\Delta^{(2)}_{\ell m}(\mathbf{k},\eta_0)}{2\ell+1} = \int_0^{\eta_0} d\eta\, e^{-\tau(\eta)}\sum_{\ell_2} S_{\ell_2 m}\, j_\ell^{(l_2,m)}[k(\eta_0-\eta)]\,, \qquad (6.30)$$

where $j_\ell^{(l_2,m)}[k(\eta_0-\eta)]$ are the so called radial functions. Of course the main information at second order is included in the source term containing different effects due to the non-linearity of the perturbations. In the total angular momentum method of Refs. [34, 36] Eq. (6.30) is interpreted just as the intergration over the radial coordinate $(\chi=\eta_0-\eta)$ of the projected source term. Another important comment is that, as in linear theory, the integral solution (6.28) is in fact just a formal solution, since the source term S contains itself the second-order photon moments up to $l=2$ (see Eq. (6.18)). This means that one has anyway to resort to the hierarchy equations for photons, Eq. (6.17), to solve for these moments. Neverthless, as in linear theory [35], one expects to need just a few moments beyond $\ell=2$ in the hierarchy equations, and once the moments entering in the source function are computed the higher moments are obtained from the integral solution. Thus the integral solution should in fact be more advantageous than solving the system of coupled equations (6.17).

7. The Boltzmann equation for baryons and cold dark matter

In this section we will derive the Boltzmann equation for massive particles, which is the case of interest for baryons and dark matter. These equations are necessary to find the time evolution of number densities and velocities of the baryon fluid which appear in the brightness equation, thus allowing to close the system of equations. Let us start from the baryon component. Electrons are tightly coupled to protons via Coulomb interactions. This forces the relative energy density contrasts and the velocities to a common value, $\delta_e = \delta_p \equiv \delta_b$ and $\mathbf{v}_e = \mathbf{v}_p \equiv \mathbf{v}$, so that we can identify electrons and protons collectively as "baryonic" matter.

To derive the Boltzmann equation for baryons let us first focus on the collisionless equation and compute therefore $dg/d\eta$, where g is the distribution function for a massive species with mass m. One of the differences with respect to photons is just that baryons are non-relativistic for the epochs of interest. Thus the first step is to generalize the formulae in Section 4 up to Eq. (4.21) to the case of a massive particle. In this case one enforces the constraint $Q^2 = g_{\mu\nu}Q^\mu Q^\nu = -m^2$ and it also useful to use the particle energy $E = \sqrt{q^2 + m^2}$, where q is defined as in Eq. (4.3). Moreover in this case it is very convenient to take the distribution function as a function of the variables $q^i = qn^i$, the position x^i and time η, without using the explicit splitting into the magnitude of the momentum q (or the energy E) and its direction n^i. Thus the total time derivative of the distribution functions reads

$$\frac{dg}{d\eta} = \frac{\partial g}{\partial \eta} + \frac{\partial g}{\partial x^i}\frac{dx^i}{d\eta} + \frac{\partial g}{\partial q^i}\frac{dq^i}{d\eta} \, . \tag{7.1}$$

We will not give the details of the calculation since we just need to replicate the same computation we did for the photons. For the four-momentum of the particle notice that Q^i has the same form as Eq. (4.6), while for Q^0 we find

$$Q^0 = \frac{e^{-\Phi}}{a} \, E \left(1 + \omega_i \frac{q^i}{E}\right) \, . \tag{7.2}$$

In the following we give the expressions for $dx^i/d\eta$ and $dq^i/d\eta$.
a) As in Eq. (4.10) $dx^i/d\eta = Q^i/Q^0$ and it turns out to be

$$\frac{dx^i}{d\eta} = \frac{q}{E} n^i e^{\Phi+\Psi} \left(1 - \omega_i n^i \frac{q}{E}\right) \left(1 - \frac{1}{2}\chi_{km} n^k n^m\right) \, . \tag{7.3}$$

b) For $dq^i/d\eta$ we need the expression of Q^i which is the same as that of Eq. (4.6)

$$Q^i = \frac{q^i}{a} e^{\Psi} \left(1 - \frac{1}{2}\chi_{km} n^k n^m\right) \, . \tag{7.4}$$

The spatial component of the geodesic equation, up to second order, reads

$$\begin{aligned}
\frac{dQ^i}{d\eta} &= -2(\mathcal{H}-\Psi')\left(1-\frac{1}{2}\chi_{km}n^k n^m\right)\frac{q}{a}n^i e^{\Psi} + e^{\Phi+2\Psi} \\
&\times \left(\frac{\partial\Psi}{\partial x^k}\frac{q^2}{aE}(2n^i n^k - \delta^{ik}) - \frac{\partial\Phi}{\partial x^i}\frac{E}{a}\right) - \frac{E}{a}\big[\omega^{i\prime} + \mathcal{H}\omega^i + q^k\big(\chi^{i\prime}_{k} \\
&+ \omega^i_{,k} - \omega^{,i}_{k}\big)\big] + \left[\mathcal{H}\omega^i\delta_{jk} - \frac{1}{2}(\chi^i{}_{j,k} + \chi^i{}_{k,j} - \chi_{jk}^{,i})\right]\frac{q^j q^k}{Ea}\,. \quad (7.5)
\end{aligned}$$

Proceeding as in the massless case we now take the total time derivative of Eq. (7.4) and using Eq. (7.5) we find

$$\begin{aligned}
\frac{dq^i}{d\eta} &= -(\mathcal{H}-\Psi')q^i + \Psi_{,k}\frac{q^i q^k}{E}e^{\Phi+\Psi} - \Phi^{,i}Ee^{\Phi+\Psi} - \Psi_{,i}\frac{q^2}{E}e^{\Phi+\Psi} \\
&- E(\omega^{i\prime} + \mathcal{H}\omega^i) - (\chi^{i\prime}_{k} + \omega^i_{,k} - \omega^{,i}_{k})Eq^k \\
&+ \left[\mathcal{H}\omega^i\delta_{jk} - \frac{1}{2}(\chi^i{}_{j,k} + \chi^i{}_{k,j} - \chi_{jk}^{,i})\right]\frac{q^j q^k}{E}\,. \quad (7.6)
\end{aligned}$$

We can now write the total time derivative of the distribution function as

$$\begin{aligned}
\frac{dg}{d\eta} &= \frac{\partial g}{\partial\eta} + \frac{q}{E}n^i e^{\Phi+\Psi}\left(1 - \omega_i n^i - \frac{1}{2}\chi_{km}n^k n^m\right)\frac{\partial g}{\partial x^i} \\
&+ \Bigg[-(\mathcal{H}-\Psi')q^i + \Psi_{,k}\frac{q^i q^k}{E}e^{\Phi+\Psi} - \Phi^{,i}Ee^{\Phi+\Psi} - \Psi_{,i}\frac{q^2}{E}e^{\Phi+\Psi} \\
&- E(\omega^{i\prime} + \mathcal{H}\omega^i) - (\chi^{i\prime}_{k} + \omega^i_{,k} - \omega^{,i}_{k})Eq^k \\
&+ \left(\mathcal{H}\omega^i\delta_{jk} - \frac{1}{2}(\chi^i{}_{j,k} + \chi^i{}_{k,j} - \chi_{jk}^{,i})\right)\frac{q^j q^k}{E}\Bigg]\frac{\partial g}{\partial q^i}\,. \quad (7.7)
\end{aligned}$$

This equation is completely general since we have just solved for the kinematics of massive particles. As far as the collision terms are concerned, for the system of electrons and protons we consider the Coulomb scattering processes between the electrons and protons and the Compton scatterings between photons and electrons

$$\frac{dg_e}{d\eta}(\mathbf{x},\mathbf{q},\eta) = \langle c_{ep}\rangle_{QQ'q'} + \langle c_{e\gamma}\rangle_{pp'q'} \quad (7.8)$$

$$\frac{dg_p}{d\eta}(\mathbf{x},\mathbf{Q},\eta) = \langle c_{ep}\rangle_{qq'Q'}\,, \quad (7.9)$$

where we have adopted the same formalism of Ref. [28] with $\mathbf{p}$ and $\mathbf{p}'$ the initial and final momenta of the photons, $\mathbf{q}$ and $\mathbf{q}'$ the corresponding quantities for the electrons and for protons $\mathbf{Q}$ and $\mathbf{Q}'$. The integral over different momenta is indicated by

$$\langle \cdots \rangle_{pp'q'} \equiv \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3p'}{(2\pi)^3} \int \frac{d^3q'}{(2\pi)^3} \cdots , \tag{7.10}$$

and thus one can read $c_{e\gamma}$ as the unintegrated part of Eq. (5.2), and similarly for c_{ep} (with the appropriate amplitude $|M|^2$). In Eq. (7.8) Compton scatterings between protons and photons can be safely neglected because the amplitude of this process has a much smaller amplitude than Compton scatterings with electrons being weighted by the inverse squared mass of the particles.

At this point for the photons we considered the perturbations around the zero-order Bose-Einstein distribution function (which are the unknown quantities). For the electrons (and protons) we can take the thermal distribution described by Eq. (5.4). Moreover we will take the moments of Eqs. (7.8)-(7.9) in order to find the energy-momentum continuity equations.

7.1. Energy continuity equations

We now integrate Eq. (7.7) over $d^3q/(2\pi)^3$. Let us recall that in terms of the distribution function the number density n_e and the bulk velocity $\mathbf{v}$ are given by

$$n_e = \int \frac{d^3q}{(2\pi)^3}\, g \,, \tag{7.11}$$

and

$$v^i = \frac{1}{n_e} \int \frac{d^3q}{(2\pi)^3}\, g\, \frac{qn^i}{E} \,, \tag{7.12}$$

where one can set $E \simeq m_e$ since we are considering non-relativistic particles. We will also make use of the following relations when integrating over the solid angle $d\Omega$

$$\int d\Omega\, n^i = \int d\Omega\, n^i n^j n^k = 0 \,, \quad \int \frac{d\Omega}{4\pi}\, n^i n^j = \frac{1}{3}\delta^{ij} \,. \tag{7.13}$$

Finally notice that $dE/dq = q/E$ and $\partial g/\partial q = (q/E)\partial g/\partial E$.

Thus the first two integrals just brings n_e' and $(n_e v^i)_{,i}$. Notice that all the terms proportional to the second-order vector and tensor perturbations of the metric give a vanishing contribution at second order since in this case we can take the zero-order distribution functions which depends only on η and E, integrate over the direction and use the fact that $\delta^{ij}\chi_{ij}=0$. The trick to solve the remaining integrals is an integration by parts over q^i. We have an integral like (the one multiplying $(\Psi'-\mathcal{H})$)

$$\int \frac{d^3q}{(2\pi)^3} q^i \frac{\partial g}{\partial q^i} = -3\int \frac{d^3q}{(2\pi)^3} g = -3n_e\,, \tag{7.14}$$

after an integration by parts over q^i. The remaining integrals can be solved still by integrating by parts over q^i. The integral proportional to $\Phi^{,i}$ in Eq. (7.7) gives

$$\int \frac{d^3q}{(2\pi)^3} E = -v_i\, n_e\,, \tag{7.15}$$

where we have used the fact that $dE/dq^i = q^i/E$. For the integral

$$\int \frac{d^3q}{(2\pi)^3} \frac{q^i q^k}{E} \frac{\partial g}{\partial q^i}\,, \tag{7.16}$$

the integration by parts brings two pieces, one from the derivation of q^iq^k and one from the derivation of the energy E

$$-4\int \frac{d^3q}{(2\pi)^3} g\frac{q^k}{E} + \int \frac{d^3q}{(2\pi)^3} g\frac{q^2}{E}\frac{q^k}{E} = -4v^k\, n_e + \int \frac{d^3q}{(2\pi)^3} g\frac{q^2}{E^2}\frac{q^k}{E}\,. \tag{7.17}$$

The last integral in Eq. (7.17) can indeed be neglected. To check this one makes use of the explicit expression (5.4) for the distribution function g to derive

$$\frac{\partial g}{\partial v^i} = g\frac{q_i}{T_e} - \frac{m_e}{T_e} v_i g\,, \tag{7.18}$$

and

$$\int \frac{d^3q}{(2\pi)^3} g q^i q^j = \delta^{ij} n_e m_e T_e + n_e m_e^2 v^i v^j\,. \tag{7.19}$$

Thus it is easy to compute

$$\frac{\Psi_{,k}}{m_e^3}\int \frac{d^3q}{(2\pi)^3} g q^2 q^k = -\Psi_{,k} v^2 \frac{T_e}{m_e} + 3\Psi_{,k} v_k n_e \frac{T_e}{m_e} + \Psi_{,k} v_k v^2\,, \tag{7.20}$$

which is negligible taking into account that T_e/m_e is of the order of the thermal velocity squared.

With these results we are now able to compute the left-hand side of the Boltzmann equation (7.8) integrated over $d^3q/(2\pi)^3$. The same operation must be done for the collision terms on the right hand side. For example for the first of the equations in (7.8) this brings to the integrals $\langle c_{ep}\rangle_{QQ'qq'} + \langle c_{e\gamma}\rangle_{pp'qq'}$.

However looking at Eq. (5.3) one realizes that $\langle c_{e\gamma}\rangle_{pp'qq'}$ vanishes because the integrand is antisymmetric under the change $\mathbf{q} \leftrightarrow \mathbf{q}'$ and $\mathbf{p} \leftrightarrow \mathbf{p}'$. In fact this is simply a consequence of the fact that the electron number is conserved for this process. The same argument holds for the other term $\langle c_{ep}\rangle_{QQ'qq'}$. Therefore the right-hand side of Eq. (7.8) integrated over $d^3q/(2\pi)^3$ vanishes and we can give the evolution equation for n_e. Collecting the results of Eq. (7.14) to (7.20) we find

$$\frac{\partial n_e}{\partial \eta} + e^{\Phi+\Psi}\frac{\partial (v^i n_e)}{\partial x^i} + 3(\mathcal{H}-\Psi')n_e + e^{\Phi+\Psi}v^k n_e\left(\Phi_{,k} - 2\Psi_{,k}\right) = 0\,. \tag{7.21}$$

Similarly, for CDM particles, we find

$$\begin{aligned}\frac{\partial n_{\rm CDM}}{\partial \eta} &+ e^{\Phi+\Psi}\frac{\partial (v^i n_{\rm CDM})}{\partial x^i} + 3(\mathcal{H}-\Psi')n_{\rm CDM} \\ &+ e^{\Phi+\Psi}v^k_{\rm CDM}\, n_{\rm CDM}\left(\Phi_{,k} - 2\Psi_{,k}\right) = 0\,. \end{aligned}\tag{7.22}$$

7.2. Momentum continuity equations

Let us now multiply Eq. (7.7) by $(q^i/E)/(2\pi)^3$ and integrate over d^3q. In this way we will find the continuity equation for the momentum of baryons. The first term just gives $(n_e v^i)'$. The second integral is of the type

$$\frac{\partial}{\partial x^j}\int\frac{d^3q}{(2\pi)^3}\,g\,\frac{qn^j}{E}\frac{qn^i}{E} = \frac{\partial}{\partial x^j}\left(n_e\frac{T_e}{m_e}\delta^{ij} + n_e v^i v^j\right)\,, \tag{7.23}$$

where we have used Eq. (7.19) and $E = m_e$. The third term proportional to $(\mathcal{H}-\Psi')$ is

$$\int\frac{d^3q}{(2\pi)^3}q^k\frac{\partial g}{\partial q_k}\frac{q^i}{E} = 4n_e + \int\frac{d^3q}{(2\pi)^3}\,g\,\frac{q^2}{E^2}\frac{q^i}{E}\,, \tag{7.24}$$

where we have integrated by parts over q^i. Notice that the last term in Eq. (7.24) is negligible being the same integral we discussewd above in Eq. (7.20). By the same arguments that lead to neglect the term of Eq. (7.20) it is easy to check that all the remaining integrals proportional to the gravitational potentials are negligible except for

$$-\,e^{\Phi+\Psi}\Phi_{,k}\int\frac{d^3q}{(2\pi)^3}\frac{\partial g}{\partial q_k}q^i = n_e e^{\Phi+\Psi}\Phi^{,i}\,. \tag{7.25}$$

The integrals proportional to the second-order vector and tensor perturbations vanish as vector and tensor perturbations are traceless and divergence-free. The only one which survives is the term proportional to $\omega^{i\prime} + \mathcal{H}\omega^i$ in Eq. (7.7).

Therefore for the integral over d^3qq^i/E of the left-hand side of the Boltzmann equation (7.7) for a massive particle with mass m_e (m_p) and distribution function (5.4) we find

$$\int \frac{d^3q}{(2\pi)^3}\frac{q^i}{E}\frac{dg_e}{d\eta} = \frac{\partial(n_e v^i)}{\partial\eta} + 4(\mathcal{H}-\Psi')n_e v^i + \Phi^{,i}e^{\Phi+\Psi}n_e$$
$$+e^{\Phi+\Psi}\left(n_e\frac{T_e}{m_e}\right)^{,i} + e^{\Phi+\Psi}\frac{\partial}{\partial x^j}(n_e v^j v^i) + \frac{\partial\omega^i}{\partial\eta}n_e + \mathcal{H}\omega^i n_e\,. \tag{7.26}$$

Now, in order to derive the momentum conservation equation for baryons, we take the first moment of both Eq. (7.8) and (7.9) multiplying them by $\mathbf{q}$ and $\mathbf{Q}$ respectively and integrating over the momenta. Since previously we integrated the left-hand side of these equations over d^3qq^i/E, we just need to multiply the previous integrals by m_e for the electrons and for m_p for the protons. Therefore if we sum the first moment of Eqs. (7.8) and (7.9) the dominant contribution on the left-hand side will be that of the protons

$$\int \frac{d^3Q}{(2\pi)^3}Q^i\,\frac{dg_p}{d\eta} = \langle c_{ep}(q^i+Q^i)\rangle_{QQ'qq'} + \langle c_{e\gamma}q^i\rangle_{pp'qq'}\,. \tag{7.27}$$

Notice that the integral of the Coulomb collision term $c_{ep}(q^i+Q^i)$ over all momenta vanishes simply because of momentum conservation (due to the Dirac function $\delta^4(q+Q-q'-Q')$). As far as the Compton scattering is concerned we have that, following Ref. [28],

$$\langle c_{e\gamma}q^i\rangle_{pp'qq'} = -\langle c_{e\gamma}p^i\rangle_{pp'qq'}\,, \tag{7.28}$$

still because of the total momentum conservation. Therefore what we can compute now is the integral over all momenta of $c_{e\gamma}p^i$. Notice however that this is equivalent just to multiply the Compton collision term $C(\mathbf{p})$ of Eq. (5.3) by p^i and integrate over $d^3p/(2\pi^3)$

$$\langle c_{e\gamma}p^i\rangle_{pp'qq'} = ae^{\Phi}\int \frac{d^3p}{(2\pi)^3}p^iC(\mathbf{p})\,. \tag{7.29}$$

where $C(\mathbf{p})$ has been already computed in Eqs. (5.39) and (5.40).

We will do the integral (7.29) in the following. First let us introduce the definition of the velocity of photons in terms of the distribution function

$$(\rho_\gamma+p_\gamma)v^i_\gamma = \int \frac{d^3p}{(2\pi)^3}fp^i\,, \tag{7.30}$$

where $p_\gamma = \rho_\gamma/3$ is the photon pressure and ρ_γ the energy density. At first order we get

$$\frac{4}{3}v_\gamma^{(1)i} = \int \frac{d\Omega}{4\pi}\Delta^{(1)}\, n^i\,, \tag{7.31}$$

where Δ is the photon distribution anisotropies defined in Eq. (6.9). At second order we instead find

$$\frac{4}{3}\frac{v_\gamma^{(2)i}}{2} = \frac{1}{2}\int \frac{d\Omega}{4\pi}\Delta^{(2)}\, n^i - \frac{4}{3}\delta_\gamma^{(1)} v_\gamma^{(1)i}\,. \tag{7.32}$$

Therefore the terms in Eqs. (5.39) and (5.40) proportional to $f^{(1)}(\mathbf{p})$ and $f^{(2)}(\mathbf{p})$ will give rise to terms containing the velocity of the photons. On the other hand the terms proportional to $f_0^{(1)}(p)$ and $f_{00}^{(2)}(p)$, once integrated, vanish because of the integral over the momentum direction n^i, $\int d\Omega n^i = 0$. Also the integrals involving $P_2(\hat{\mathbf{v}}\cdot\mathbf{n}) = [3(\hat{\mathbf{v}}\cdot\mathbf{n})^2 - 1]/2$ in the first line of Eq. (5.39) and (5.40) vanish since

$$\int d\Omega P_2(\hat{\mathbf{v}}\cdot\mathbf{n})\, n^i = \hat{v}^k\hat{v}^j \int d\Omega n_k n_j n^i = 0\,, \tag{7.33}$$

where we are using the relations (7.13). Similarly all the terms proportional to v, $(\mathbf{v}\cdot\mathbf{n})^2$ and v^2 do not give any contribution to Eq. (7.29) and, in the second-order collision term, one can check that $\int d\Omega Y_2(\mathbf{n}) n^i = 0$. Then there are terms proportional to $(\mathbf{v}\cdot\mathbf{n})f^{(0)}(p)$, $(\mathbf{v}\cdot\mathbf{n})p\partial f^{(0)}/\partial p$ and $(\mathbf{v}\cdot\mathbf{n})p\partial f_0^{(1)}/\partial p$ for which we can use the rules (6.10) when integrating over p while the integration over the momentum direction is

$$\int \frac{d\Omega}{4\pi}(\mathbf{v}\cdot\mathbf{n})n^i = v_k \int \frac{d\Omega}{4\pi} n^k n^i = \frac{1}{3}v^i\,. \tag{7.34}$$

Finally from the second line of Eq. (5.40) we get three integrals. One is

$$\int \frac{d^3p}{(2\pi)^3} p^i\,(\mathbf{v}\cdot\mathbf{n}) f^{(1)}(\mathbf{p}) = \bar{\rho}_\gamma \int \frac{d\Omega}{4\pi}\Delta^{(1)}(\mathbf{v}\cdot\mathbf{n})n^i\,, \tag{7.35}$$

where $\bar{\rho}_\gamma$ is the background energy density of the photons. The second comes from

$$\frac{1}{2}\int \frac{d^3p}{(2\pi)^3} p^i\,(\mathbf{v}\cdot\mathbf{n})P_2(\hat{\mathbf{v}}\cdot\mathbf{n})\left(f_2^{(1)}(p) - p\frac{\partial f_2^{(1)}(p)}{\partial p}\right) \tag{7.36}$$

$$= \frac{5}{4}\bar{\rho}_\gamma\Delta_2^{(1)}\left[3v_j\hat{v}_k\hat{v}_l\int\frac{d\Omega}{4\pi}n^i n^j n^k n^l - v_j\int\frac{d\Omega}{4\pi}n^i n^j\right] = \frac{1}{3}\bar{\rho}_\gamma\Delta_2^{(1)}\hat{v}^i\,,$$

where we have used the rules (6.10), Eq. (7.13) and $\int(d\Omega/4\pi)\, n^i n^j n^k n^l = (\delta^{ij}\delta^{kl}+\delta^{ik}\delta^{lj}+\delta^{il}\delta^{jk})/15$. In fact the third integral

$$-\int \frac{d^3p}{(2\pi)^3} p^i (\mathbf{v}\cdot\mathbf{n}) f_2^{(1)}(p)\,, \tag{7.37}$$

exactly cancels the previous one. Summing the various integrals we find

$$\int \frac{d\mathbf{p}}{(2\pi)^3} C(\mathbf{p})\mathbf{p} = n_e \sigma_T \bar{\rho}_\gamma \Bigg[\frac{4}{3}(\mathbf{v}^{(1)}-\mathbf{v}_\gamma^{(1)}) - \int \frac{d\Omega}{4\pi}\frac{\Delta^{(2)}}{2}\mathbf{n} + \frac{4}{3}\frac{\mathbf{v}^{(2)}}{2} + \frac{4}{3}\delta_e^{(1)}(\mathbf{v}^{(1)}-\mathbf{v}_\gamma^{(1)}) + \int \frac{d\Omega}{4\pi}\Delta^{(1)}(\mathbf{v}\cdot\mathbf{n})\mathbf{n} + \Delta_0^{(1)}\mathbf{v}\Bigg]\,. \tag{7.38}$$

Eq. (7.38) can be further simplified. Recalling that $\delta_\gamma^{(1)} = \Delta_0^{(1)}$ we use Eq. (7.32) and notice that

$$\int \frac{d\Omega}{4\pi}\Delta^{(1)}\,(\mathbf{v}\cdot\mathbf{n})n^i = v_j^{(1)}\Pi_\gamma^{ji} + \frac{1}{3}v^i\Delta_0^{(1)}\,, \tag{7.39}$$

where the photon quadrupole Π_γ^{ij} is defined as

$$\Pi_\gamma^{ij} = \int \frac{d\Omega}{4\pi}\left(n^i n^j - \frac{1}{3}\delta^{ij}\right)\left(\Delta^{(1)} + \frac{\Delta^{(2)}}{2}\right)\,. \tag{7.40}$$

Thus, our final expression for the integrated collision term (7.29) reads

$$\int \frac{d^3p}{(2\pi)^3} C(\mathbf{p})p^i = n_e\sigma_T\bar{\rho}_\gamma\left[\frac{4}{3}(v^{(1)i}-v_\gamma^{(1)i}) + \frac{4}{3}\left(\frac{v^{(2)i}}{2} - \frac{v_\gamma^{(2)i}}{2}\right) + \frac{4}{3}\left(\delta_e^{(1)} + \Delta_0^{(1)}\right)(v^{(1)i}-v_\gamma^{(1)i}) + v_j^{(1)}\Pi_\gamma^{ji}\right]\,. \tag{7.41}$$

We are now able to give the momentum continuity equation for baryons by combining $m_p dg_p/d\eta$ from Eq. (7.26) with the collision term (7.29)

$$\begin{aligned}\frac{\partial(\rho_b v^i)}{\partial\eta} &+ 4(\mathcal{H}-\Psi')\rho_b v^i + \Phi^{,i}e^{\Phi+\Psi}\rho_b + e^{\Phi+\Psi}\left(\rho_b\frac{T_b}{m_p}\right)^{,i}\\ &+ e^{\Phi+\Psi}\frac{\partial}{\partial x^j}(\rho_b v^j v^i) + \frac{\partial\omega^i}{\partial\eta}\rho_b + \mathcal{H}\omega^i\rho_b\\ &= -n_e\sigma_T a\,\bar{\rho}_\gamma\left[\frac{4}{3}(v^{(1)i}-v_\gamma^{(1)i}) + \frac{4}{3}\left(\frac{v^{(2)i}}{2}-\frac{v_\gamma^{(2)i}}{2}\right)\right.\\ &+ \left.\frac{4}{3}\left(\delta_b^{(1)}+\Delta_0^{(1)}+\Phi^{(1)}\right)(v^{(1)i}-v_\gamma^{(1)i}) + v_j^{(1)}\Pi_\gamma^{ji}\right]\,,\end{aligned} \tag{7.42}$$

where ρ_b is the baryon energy density and, as we previously explained, we took into account that to a good approximation the electrons do not contribute to the mass of baryons. In the following we will expand explicitly at first and second-order Eq. (7.42).

7.2.1. First-order momentum continuity equation for baryons
At first order we find

$$\frac{\partial v^{(1)i}}{\partial\eta}+\mathcal{H}v^{(1)i}+\Phi^{(1),i}=\frac{4}{3}\tau'\frac{\bar{\rho}_\gamma}{\bar{\rho}_b}\left(v^{(1)i}-v_\gamma^{(1)i}\right)\,. \tag{7.43}$$

7.2.2. Second-order momentum continuity equation for baryons
At second order there are various simplifications. In particular notice that the term on the right-hand side of Eq. (7.42) which is proportional to δ_b vanishes when matched to expansion of the left-hand side by virtue of the first-order equation (7.43). Thus, at the end we find a very simple equation

$$\begin{aligned}&\frac{1}{2}\left(\frac{\partial v^{(2)i}}{\partial\eta}+\mathcal{H}v^{(2)i}+2\frac{\partial\omega^i}{\partial\eta}+2\mathcal{H}\omega_i+\Phi^{(2),i}\right)-\frac{\partial\Psi^{(1)}}{\partial\eta}v^{(1)i}\\&+v^{(1)j}\partial_j v^{(1)i}+(\Phi^{(1)}+\Psi^{(1)})\Phi^{(1),i}+\left(\frac{T_b}{m_p}\right)^{,i}=\frac{4}{3}\tau'\frac{\bar{\rho}_\gamma}{\bar{\rho}_b}\\&\times\left[\left(\frac{v^{(2)i}}{2}-\frac{v_\gamma^{(2)i}}{2}\right)+\left(\Delta_0^{(1)}+\Phi^{(1)}\right)\left(v^{(1)i}-v_\gamma^{(1)i}\right)+\frac{3}{4}v_j^{(1)}\Pi_\gamma^{ji}\right]\,,\end{aligned} \tag{7.44}$$

with $\tau'=-\bar{n}_e\sigma_T a$.

7.2.3. First-order momentum continuity equation for CDM
Since CDM particles are collisionless, at first order we find

$$\frac{\partial v^{(1)i}_{\rm CDM}}{\partial\eta}+\mathcal{H}v^{(1)i}_{\rm CDM}+\Phi^{(1),i}=0\,. \tag{7.45}$$

7.2.4. Second-order momentum continuity equation for CDM
At second order we find

$$\begin{aligned}&\frac{1}{2}\left(\frac{\partial v^{(2)i}_{\rm CDM}}{\partial\eta}+\mathcal{H}v^{(2)i}_{\rm CDM}+2\frac{\partial\omega^i}{\partial\eta}+2\mathcal{H}\omega_i+\Phi^{(2),i}\right)-\frac{\partial\Psi^{(1)}}{\partial\eta}v^{(1)i}_{\rm CDM}\\&+v^{(1)j}_{\rm CDM}\,\partial_j v^{(1)i}_{\rm CDM}+(\Phi^{(1)}+\Psi^{(1)})\Phi^{(1),i}+\left(\frac{T_{\rm CDM}}{m_{\rm CDM}}\right)^{,i}=0\,.\end{aligned} \tag{7.46}$$

8. Linear solution of the Boltzmann equations

In this section we will solve the Boltzmann equations at first order in perturbation theory. The interested reader will find the extension of these formulae to second order in Ref. [2]. The first two moments of the photon Boltzmann equation are obtained by integrating Eq. (6.6) over $d\Omega_{\mathbf{n}}/4\pi$ and $d\Omega_{\mathbf{n}} n^i/4\pi$ respectively and they lead to the density and velocity continuity equations

$$\Delta_{00}^{(1)'} + \frac{4}{3}\partial_i v_\gamma^{(1)i} - 4\Psi^{(1)'} = 0\,, \tag{8.1}$$

$$v_\gamma^{(1)i\prime} + \frac{3}{4}\partial_j \Pi_\gamma^{(1)ji} + \frac{1}{4}\Delta_{00}^{(1),i} + \Phi^{(1),i} = -\tau' \left(v^{(1)i} - v_\gamma^{(1)} \right)\,, \tag{8.2}$$

where Π^{ij} is the photon quadrupole moment, defined in Eq. (7.40).

Let us recall here that $\delta_\gamma^{(1)} = \Delta_{00}^{(1)} = \int d\Omega \Delta^{(1)}/4\pi$ and that the photon velocity is given by Eq. (7.31).

The two equations above are complemented by the momentum continuity equation for baryons, which can be conveniently written as

$$v^{(1)i} = v_\gamma^{(1)i} + \frac{R}{\tau'}\left[v^{(1)i\prime} + \mathcal{H} v^{(1)i} + \Phi^{(1),i} \right]\,, \tag{8.3}$$

where we have introduced the baryon-photon ratio $R \equiv 3\rho_b/(4\rho_\gamma)$.

Eq. (8.3) is in a form ready for a consistent expansion in the small quantity τ^{-1} which can be performed in the tight-coupling limit. By first taking $v^{(1)i} = v_\gamma^{(1)i}$ at zero order and then using this relation in the left-hand side of Eq. (8.3) one obtains

$$v^{(1)i} - v_\gamma^{(1)i} = \frac{R}{\tau'}\left[v_\gamma^{(1)i\prime} + \mathcal{H} v_\gamma^{(1)i} + \Phi^{(1),i} \right]\,. \tag{8.4}$$

Such an expression for the difference of velocities can be used in Eq. (8.2) to give the evolution equation for the photon velocity in the limit of tight coupling

$$v_\gamma^{(1)i\prime} + \mathcal{H}\frac{R}{1+R} v_\gamma^{(1)i} + \frac{1}{4}\frac{\Delta_{00}^{(1),i}}{1+R} + \Phi^{(1),i} = 0\,. \tag{8.5}$$

Notice that in Eq. (8.5) we are neglecting the quadrupole of the photon distribution $\Pi^{(1)ij}$ (and all the higher moments) since it is well known that at linear order such moment(s) are suppressed in the tight-coupling limit by (successive powers of) $1/\tau$ with respect to the first two moments, the photon energy density and velocity. Eqs. (8.1) and (8.5) are the master equations which govern the photon-baryon fluid acoustic oscillations before the epoch of recombination when photons and baryons are tightly coupled by Compton scattering.

In fact one can combine these two equations to get a single second-order differential equation for the photon energy density perturbations $\Delta_{00}^{(1)}$. Deriving Eq. (8.1) with respect to conformal time and using Eq. (8.5) to replace $\partial_i v_\gamma^{(1)i}$ yields

$$\left(\Delta_{00}^{(1)\prime\prime} - 4\Psi^{(1)\prime\prime}\right) + \mathcal{H}\frac{R}{1+R}\left(\Delta_{00}^{(1)\prime} - 4\Psi^{(1)\prime}\right)$$
$$-c_s^2\nabla^2\left(\Delta_{00}^{(1)} - 4\Psi^{(1)}\right) = \frac{4}{3}\nabla^2\left(\Phi^{(1)} + \frac{\Psi^{(1)}}{1+R}\right), \tag{8.6}$$

where $c_s = 1/\sqrt{3(1+R)}$ is the speed of sound of the photon-baryon fluid. Indeed, in order to solve Eq. (8.6) one needs to know the evolution of the gravitational potentials. We will come back later to the discussion of the solution of Eq. (8.6).

A useful relation we will use in the following is obtained by considering the continuity equation for the baryon density perturbation. By perturbing at first order Eq. (7.21) we obtain

$$\delta_b^{(1)\prime} + v^i_{,i} - 3\Psi^{(1)\prime} = 0\,. \tag{8.7}$$

Subtracting Eq. (8.7) form Eq. (8.1) brings

$$\Delta_{00}^{(1)\prime} - \frac{4}{3}\delta_b^{(1)\prime} + \frac{4}{3}(v_\gamma^{(1)i} - v^{(1)i})_{,i} = 0\,, \tag{8.8}$$

which implies that at lowest order in the tight-coupling approximation

$$\Delta_{00}^{(1)} = \frac{4}{3}\delta_b^{(1)}\,, \tag{8.9}$$

for adiabatic perturbations.

8.1. Linear solutions in the limit of tight coupling

In this section we briefly recall how to obtain at linear order the solutions of the Boltzmann equations (8.6). These correspond to the acoustic oscillations of the photon-baryon fluid for modes which are within the horizon at the time of recombination. It is well known that, in the variable $(\Delta_{00}^{(1)} - 4\Psi^{(1)})$, the solution can be written as [32, 38]

$$[1+R(\eta)]^{1/4}(\Delta_{00}^{(1)} - 4\Psi^{(1)}) = A\cos[kr_s(\eta)] + B\,\sin[kr_s(\eta)]$$
$$-4\frac{k}{\sqrt{3}}\int_0^\eta d\eta'[1+R(\eta')]^{3/4}\left(\Phi^{(1)}(\eta') + \frac{\Psi^{(1)}(\eta')}{1+R(\eta')}\right)$$
$$\times\sin[k(r_s(\eta) - r_s(\eta'))]\,, \tag{8.10}$$

where the sound horizon is given by $r_s(\eta) = \int_0^\eta d\eta' c_s(\eta')$, with $R = 3\rho_b/(4\rho_\gamma)$. The constants A and B in Eq. (8.10) are fixed by the choice of initial conditions.

In order to give an analytical solution that catches most of the physics underlying Eq. (8.10) and which remains at the same time very simple to treat, we will make some simplifications following Ref. [28, 39]. First, for simplicity, we are going to neglect the ratio R wherever it appears, *except* in the arguments of the varying cosines and sines, where we will treat $R = R_*$ as a constant evaluated at the time of recombination. In this way we keep track of a damping of the photon velocity amplitude with respect to the case $R = 0$ which prevents the acoustic peaks in the power-spectrum to disappear. Treating R as a constant is justified by the fact that for modes within the horizon the time scale of the ocillations is much shorter than the time scale on which R varies. If R is a constant the sound speed is just a constant $c_s = 1/\sqrt{3(1 + R_*)}$, and the sound horizon is simply $r_s(\eta) = c_s\eta$.

Second, we are going to solve for the evolutions of the perturbations in two well distinguished limiting regimes. One regime is for those perturbations which enter the Hubble radius when matter is the dominant component, that is at times much bigger than the equality epoch, with $k \ll k_{eq} \sim \eta_{eq}^{-1}$, where k_{eq} is the wavenumber of the Hubble radius at the equality epoch. The other regime is for those perturbations with much smaller wavelenghts which enter the Hubble radius when the universe is still radiation dominated, that is perturbations with wavenumbers $k \gg k_{eq} \sim \eta_{eq}^{-1}$. In fact we are interested in perturbation modes which are within the horizon by the time of recombination η_*. Therefore we will further suppose that $\eta_* \gg \eta_{eq}$ in order to study such modes in the first regime. Even though $\eta_* \gg \eta_{eq}$ is not the real case, it allows to obtain some analytical expressions.

Before solving for these two regimes let us fix our initial conditions, which are taken on large scales deep in the radiation dominated era (for $\eta \to 0$). During this epoch, for adiabatic perturbations, the gravitational potentials remain constant on large scales (we are neglecting anisotropic stresses so that $\Phi^{(1)} \simeq \Psi^{(1)}$) and from the $(0-0)$-component of Einstein equations

$$\Phi^{(1)}(0) = -\frac{1}{2}\Delta_{00}^{(1)}(0)\,. \tag{8.11}$$

On the other hand, from the energy continuity equation (8.1) on large scales

$$\Delta_{00}^{(1)} - 4\Psi^{(1)} = \mathrm{const.}\,; \tag{8.12}$$

from Eq. (8.11) the constant on the right-hand side of Eq. (8.12) is fixed to be $-6\Phi^{(1)}(0)$; thus we find $B = 0$ and $A = -6\Phi^{(1)}(0)$.

With our simplifications Eq. (8.10) then reads

$$\Delta^{(1)}_{00} - 4\Psi^{(1)} = - 6\Phi^{(1)}(0)\cos(\omega_0\eta) - \frac{8k}{\sqrt{3}}\int_0^\eta d\eta' \Phi^{(1)}(\eta')\sin[\omega_0(\eta-\eta')]\,, \tag{8.13}$$

where $\omega_0 = kc_s$.

8.2. Perturbation modes with $k \ll k_{eq}$

This regime corresponds to perturbation modes which enter the Hubble radius when the universe is matter dominated at times $\eta \gg \eta_{eq}$. During matter domination the gravitational potential remains constant (both on super-horizon and sub-horizon scales), as one can see for example from Eq. (B.1), and its value is fixed to $\Phi^{(1)}(\mathbf{k},\eta) = \frac{9}{10}\Phi^{(1)}(0)$, where $\Phi^{(1)}(0)$ corresponds to the gravitational potential on large scales during the radiation dominated epoch. Since we are interested in the photon anisotropies around the time of recombination, when matter is dominating, we can perform the integral appearing in Eq. (8.10) by taking the gravitational potential equal to its value during matter domination so that it is easily computed

$$2\int_0^\eta d\eta' \Phi^{(1)}(\eta')\sin[\omega_0(\eta-\eta')] = \frac{18}{10}\frac{\Phi^{(1)}(0)}{\omega_0}\left(1-\cos(\omega_0\eta)\right)\,. \tag{8.14}$$

Thus Eq. (8.13) gives

$$\Delta^{(1)}_{00} - 4\Psi^{(1)} = \frac{6}{5}\Phi^{(1)}(0)\cos(\omega_0\eta) - \frac{36}{5}\Phi^{(1)}(0)\,. \tag{8.15}$$

The baryon-photon fluid velocity can then be obtained as $\partial_i v^{(1)i}_\gamma = -3(\Delta^{(1)}_{00} - 4\Psi^{(1)})'/4$ from Eq. (8.1). In Fourier space

$$ik_i\, v^{(1)i}_\gamma = \frac{9}{10}\Phi^{(1)}(0)\sin(\omega_0\eta)\omega_0\,, \tag{8.16}$$

where, going to Fourier space, $\partial_i v^{(1)i}_\gamma \to ik_i\, v^{(1)i}_\gamma(\mathbf{k})$ and

$$v^{(1)i}_\gamma = -i\frac{k^i}{k}\frac{9}{10}\Phi^{(1)}(0)\sin(\omega_0\eta)c_s\,, \tag{8.17}$$

since the linear velocity is irrotational.

8.3. Perturbation modes with $k \gg k_{eq}$

This regime corresponds to perturbation modes which enter the Hubble radius when the universe is still radiation dominated at times $\eta \ll \eta_{eq}$. In this case an approximate analytical solution for the evolution of the perturbations can be obtained by considering the gravitational potential for a pure radiation dominated epoch, given by Eq. (B.8). For the integral in Eq. (8.13) we thus find

$$\int_0^\eta \Phi^{(1)}(\eta') \sin[\omega_0(\eta - \eta')] = -\frac{3}{2\omega_0} \cos(\omega_0 \eta) \,, \tag{8.18}$$

where we have kept only the dominant contribution oscillating in time, while neglecting terms which decay in time. The solution (8.13) becomes

$$\Delta^{(1)}_{00} - 4\Psi^{(1)} = 6\Phi^{(1)}(0) \cos(\omega_0 \eta) \,, \tag{8.19}$$

and the velocity is given by

$$v_\gamma^{(1)i} = -i \frac{k^i}{k} \frac{9}{2} \Phi^{(1)}(0) \sin(\omega_0 \eta) c_s \,, \tag{8.20}$$

Notice that the solutions (8.19)–(8.20) are actually correct only when radiation dominates. Indeed, between the epoch of equality and recombination, matter starts to dominate. Full account of such a period is given e.g. in Section 7.3 of Ref. [28], while its consequences for the CMB anisotropy evolution can be found e.g. in Ref. [40].

9. Conclusions

In these lecture notes we derived the equations which allow to evaluate CMB anisotropies, by computing the Boltzmann equations describing the evolution of the baryon-photon fluid up to second order. This allows to follow the time evolution of CMB anisotropies (up to second order) on all scales, from the early epoch, when the cosmological perturbations were generated, to the present time, through the recombination era. The dynamics at second order is particularly important when dealing with the issue of non-Gaussianity in CMB anisotropies. Indeed, many mechanisms for the generation of the primordial inhomogeneities predict a level of non-Gaussianity in the curvature perturbation which might be detectable by present and future experiments.

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Symbol	Definition	Equation
Φ, Ψ	Gravitational potentials in Poisson gauge	(3.1)
ω_i	2nd-order vector perturbation in Poisson gauge	(3.1)
χ_{ij}	2nd-order tensor perturbation in Poisson gauge	(3.1)
η	Conformal time	(3.1)
f	Photon distribution function	(4.8)
g	Distribution function for massive particles	(5.4) & (7.1)
$f^{(i)}$	i-th order perturbation of the photon distribution function	(4.23)
$f^{(i)}_{\ell m}$	Moments of the photon distribution function	(5.17)
$C(\mathbf{p})$	Collision term	(5.3) & (5.7)
p	Magnitude of photon momentum ($\mathbf{p} = pn^i$)	(4.3)
n^i	Propagation direction	(4.6)
$\Delta^{(1)}(x^i, n^i, \eta)$	First-order fractional energy photon fluctuations	(6.5)
$\Delta^{(2)}(x^i, n^i, \eta)$	Second-order fractional energy photon fluctuations	(6.9)
n_e	Electron number density	(7.11)
$\delta_e(\delta_b)$	Electron (baryon) density perturbation	(5.13)
$\mathbf{k}$	Wavenumber	(6.16)
v_m	Baryon velocity perturbation	(6.19) & (6.20)
$v^{(2)i}_{\rm CDM}$	Cold dark matter velocity	(7.46)
$v^{(2)i}_{\gamma}$	Second-order photon velocity	(7.32)
$S_{\ell m}$	Temperature source term	(6.17)
τ	Optical depth	(6.14)
$\bar{\rho}_\gamma(\bar{\rho}_b)$	Background photon (baryon) energy density	(7.43)

Appendix A. Einstein's equations

In this Appendix we provide the necessary expressions to deal with the gravitational part of the problem we are interested in, namely the second-order CMB anisotropies generated at recombination as well as the acoustic oscillations of the baryon-photon fluid. The first part of the Appendix contains the expressions for the metric, connection coefficients and Einstein's tensor perturbed up to second order around a flat Friedmann-Robertson-Walker background, the energy-momentum tensors for massless (photons) and massive (baryons and cold dark matter) particles, and the relevant Einstein's equations. The second part deals with the evolution equations and the solutions for the second-order gravitational potentials in the Poisson gauge.

Appendix A.1. The metric tensor

As discussed in Section 3, we write our second-order metric in the Poisson gauge,

$$ds^2 = a^2(\eta)\left[-e^{2\Phi}d\eta^2 + 2\omega_i dx^i d\eta + (e^{-2\Psi}\delta_{ij} + \chi_{ij})dx^i dx^j\right] \,, \qquad \text{(A.1)}$$

where $a(\eta)$ is the scale factor as a function of conformal time η, and ω_i and χ_{ij} are vector and tensor peturbation modes respectively. Each metric perturbation is expanded into a linear (first-order) and a second-order part, as discussed in Section 3.

Appendix A.2. The connection coefficients

For the connection coefficients we find

$$\begin{aligned}
\Gamma^0_{00} &= \mathcal{H} + \Phi' \, , \\
\Gamma^0_{0i} &= \frac{\partial \Phi}{\partial x^i} + \mathcal{H}\omega_i \, , \\
\Gamma^i_{00} &= \omega^{i'} + \mathcal{H}\omega^i + e^{2\Psi+2\Phi}\frac{\partial \Phi}{\partial x_i} \, , \\
\Gamma^0_{ij} &= -\frac{1}{2}\left(\frac{\partial \omega_j}{\partial x^i} + \frac{\partial \omega_i}{\partial x^j}\right) + e^{-2\Psi-2\Phi}\left(\mathcal{H} - \Psi'\right)\delta_{ij} + \frac{1}{2}\chi'_{ij} + \mathcal{H}\chi_{ij} \, , \\
\Gamma^i_{0j} &= \left(\mathcal{H} - \Psi'\right)\delta_{ij} + \frac{1}{2}\chi'_{ij} + \frac{1}{2}\left(\frac{\partial \omega_i}{\partial x^j} - \frac{\partial \omega_j}{\partial x^i}\right) \, , \\
\Gamma^i_{jk} &= -\mathcal{H}\omega^i\delta_{jk} - \frac{\partial \Psi}{\partial x^k}\delta^i{}_j - \frac{\partial \Psi}{\partial x^j}\delta^i{}_k + \frac{\partial \Psi}{\partial x_i}\delta_{jk} \\
&+ \frac{1}{2}\left(\frac{\partial \chi^i{}_j}{\partial x^k} + \frac{\partial \chi^i{}_k}{\partial x^j} - \frac{\partial \chi_{jk}}{\partial x_i}\right) \, . \qquad \text{(A.2)}
\end{aligned}$$

Appendix A.3. Einstein tensor

The components of Einstein's tensor read

$$\begin{aligned}
G^0{}_0 &= -\frac{e^{-2\Phi}}{a^2}\left[3\mathcal{H}^2 - 6\mathcal{H}\Psi' + 3(\Psi')^2\right. \\
&- \left. e^{2\Phi+2\Psi}\left(\partial_i\Psi\partial^i\Psi - 2\nabla^2\Psi\right)\right] \, , \qquad \text{(A.3)} \\
G^i{}_0 &= 2\frac{e^{2\Psi}}{a^2}\left[\partial^i\Psi' + \left(\mathcal{H} - \Psi'\right)\partial^i\Phi\right] - \frac{1}{2a^2}\nabla^2\omega^i \\
&+ \left(4\mathcal{H}^2 - 2\frac{a''}{a}\right)\frac{\omega^i}{a^2} \, , \qquad \text{(A.4)} \\
G^i{}_j &= \frac{1}{a^2}\left[e^{-2\Phi}\left(\mathcal{H}^2 - 2\frac{a''}{a} - 2\Psi'\Phi' - 3(\Psi')^2 + 2\mathcal{H}\left(\Phi' + 2\Psi'\right)\right.\right. \\
&+ \left.\left. 2\Psi''\right) + e^{2\Psi}\left(\partial_k\Phi\partial^k\Phi + \nabla^2\Phi - \nabla^2\Psi\right)\right]\delta^i_j + \frac{e^{2\Psi}}{a^2}
\end{aligned}$$

$$
\begin{aligned}
\times & \left(-\partial^i\Phi\partial_j\Phi - \partial^i\partial_j\Phi + \partial^i\partial_j\Psi - \partial^i\Phi\partial_j\Psi + \partial^i\Psi\partial_j\Psi - \partial^i\Psi\partial_j\Phi\right) \\
- & \frac{\mathcal{H}}{a^2}\left(\partial^i\omega_j + \partial_j\omega^i\right) - \frac{1}{2a^2}\left(\partial^i\omega'_j + \partial_j\omega^{i'}\right) \\
+ & \frac{1}{a^2}\left(\mathcal{H}\chi^{i'}_j + \frac{1}{2}\chi^{i''}_j - \frac{1}{2}\nabla^2\chi^i_j\right) . \qquad (A.5)
\end{aligned}
$$

Taking the traceless part of Eq. (A.5), we find $\Psi - \Phi = \mathcal{Q}$, where $\mathcal{Q}$ is defined by $\nabla^2\mathcal{Q} = -P + 3N$, with $P \equiv P^i{}_i$,

$$
P^i_j = \partial^i\Phi\partial_j\Psi + \frac{1}{2}\left(\partial^i\Phi\partial_j\Phi - \partial^i\Psi\partial_j\Psi\right) + 4\pi G_{\rm N} a^2 e^{-2\Psi} T^i_j \qquad (A.6)
$$

and $\nabla^2 N = \partial_i\partial^j P^i_j$.

The trace of Eq. (A.5) gives

$$
\begin{aligned}
& e^{-2\Phi}\left(\mathcal{H}^2 - 2\frac{a''}{a} - 2\Phi'\Psi' - 3(\Psi')^2 + 2\mathcal{H}\left(3\Psi' - \mathcal{Q}'\right) + 2\Psi''\right) \\
& + \frac{e^{2\Psi}}{3}\left(2\partial_k\Phi\partial^k\Phi + \partial_k\Psi\partial^k\Psi - 2\partial_k\Phi\partial^k\Psi + 2(P - 3N)\right) \\
& = \frac{8\pi G_{\rm N}}{3} a^2 T^k_k . \qquad (A.7)
\end{aligned}
$$

From Eq. (A.4), we may deduce an equation for ω^i

$$
\begin{aligned}
- & \frac{1}{2}\nabla^2\omega^i + \left(4\mathcal{H}^2 - 2\frac{a''}{a}\right)\omega^i \qquad (A.8) \\
= & -\left(\delta^i_j - \frac{\partial^i\partial_j}{\nabla^2}\right)\left(2\left(\partial^j\Psi' + \left(\mathcal{H} - \Psi'\right)\partial^j\Phi\right) - 8\pi G_{\rm N} a^2 e^{-2\Psi} T^j_0\right) .
\end{aligned}
$$

Appendix A.4. Energy-momentum tensor

Appendix A.4.1. Energy-momentum tensor for photons

The energy-momentum tensor for photons is defined as

$$
T^\mu_{\gamma\ \nu} = \frac{2}{\sqrt{-g}}\int \frac{d^3P}{(2\pi)^3}\,\frac{P^\mu P_\nu}{P^0}\, f\, , \qquad (A.9)
$$

where g is the determinant of the metric (A.1) and f is the distribution function. We thus obtain

$$
T^0_{\gamma\ 0} \quad = \quad -\bar{\rho}_\gamma\left(1 + \Delta^{(1)}_{00} + \frac{\Delta^{(2)}_{00}}{2}\right) , \qquad (A.10)
$$

$$T^{i}_{\gamma\ 0} = -\frac{4}{3}e^{\Psi+\Phi}\bar{\rho}_\gamma\left(v_\gamma^{(1)i}+\frac{1}{2}v_\gamma^{(2)i}+\Delta_{00}^{(1)}v_\gamma^{(1)i}\right)+\frac{1}{3}\bar{\rho}_\gamma e^{\Psi-\Phi}\omega^i \quad (A.11)$$

$$T^{i}_{\gamma\ j} = \bar{\rho}_\gamma\left(\Pi^{i}_{\gamma\ j}+\frac{1}{3}\delta^i_{\ j}\left(1+\Delta_{00}^{(1)}+\frac{\Delta_{00}^{(2)}}{2}\right)\right), \quad (A.12)$$

where $\bar{\rho}_\gamma$ is the background energy density of photons and

$$\Pi^{ij}_\gamma = \int\frac{d\Omega}{4\pi}\left(n^i n^j-\frac{1}{3}\delta^{ij}\right)\left(\Delta^{(1)}+\frac{\Delta^{(2)}}{2}\right), \quad (A.13)$$

is the quadrupole moment of the photons.

Appendix A.4.2. Energy-momentum tensor for massive particles

The energy-momentum tensor for massive particles of mass m, number density n and degrees of freedom g_d

$$T^{\mu}_{m\ \nu} = \frac{g_d}{\sqrt{-g}}\int\frac{d^3Q}{(2\pi)^3}\frac{Q^\mu Q_\nu}{Q^0}\,g_m\,, \quad (A.14)$$

where g_m is the distribution function. We obtain

$$T^{0}_{m\ 0} = -\rho_m = -\bar{\rho}_m\left(1+\delta_m^{(1)}+\frac{1}{2}\delta_m^{(2)}\right), \quad (A.15)$$

$$T^{i}_{m\ 0} = -e^{\Psi+\Phi}\rho_m v^i_m = -e^{\Phi+\Psi}\bar{\rho}_m\left(v_m^{(1)i}+\frac{1}{2}v_m^{(2)i}+\delta_m^{(1)}v_m^{(1)i}\right) \quad (A.16)$$

$$T^{i}_{m\ j} = \rho_m\left(\delta^i_{\ j}\frac{T_m}{m}+v^i_m v_{m\ j}\right) = \bar{\rho}_m\left(\delta^i_{\ j}\frac{T_m}{m}+v_m^{(1)i}v^{(1)}_{m\ j}\right), \quad (A.17)$$

where $\bar{\rho}_m$ is the background energy density of massive particles and we have included the equilibrium temperature T_m.

Appendix B. First-order solutions of Einstein's equations in various eras

Appendix B.1. Matter-dominated era

During the phase in which the CDM is dominating the energy density of the Universe, $a \sim \eta^2$ and we may use Eq. (A.7) to obtain an equation for the gravitational potential at first order in perturbation theory (for which $\Phi^{(1)} = \Psi^{(1)}$)

$$\Phi^{(1)''}+3\mathcal{H}\Phi^{(1)'} = 0\,, \quad (B.1)$$

which has two solutions $\Phi^{(1)}_{+} = \text{constant}$ and $\Phi^{(1)}_{-} = \mathcal{H}/a^2$. At the same order of perturbation theory, the CDM velocity can be read off from Eq. (A.4)

$$v^{(1)i} = -\frac{2}{3\mathcal{H}}\partial^i\Phi^{(1)}\,. \tag{B.2}$$

The matter density contrast $\delta^{(1)}$ satisfies the first-order continuity equation

$$\delta^{(1)'} = -\frac{\partial v^{(1)i}}{\partial x^i} = -\frac{2}{3\mathcal{H}}\nabla^2\Phi^{(1)}\,. \tag{B.3}$$

Going to Fourier space, this implies that

$$\delta^{(1)}_k = \delta^{(1)}_k(0) + \frac{k^2\eta^2}{6}\Phi^{(1)}_k\,, \tag{B.4}$$

where $\delta^{(1)}_k(0)$ is the initial condition in the matter-dominated period.

Appendix B.2. Radiation-dominated era

We consider a universe dominated by photons and massless neutrinos. The energy-momentum tensor for massless neutrinos has the same form as that for photons. During the phase in which radiation is dominating the energy density of the Universe, $a \sim \eta$ and we may combine Eqs. (A.3) and (A.7) to obtain an equation for the gravitational potential $\Psi^{(1)}$ at first order in perturbation theory

$$\begin{aligned}\Psi^{(1)''} + 4\mathcal{H}\Psi^{(1)'} - \frac{1}{3}\nabla^2\Psi^{(1)} &= \mathcal{H}Q^{(1)'} + \frac{1}{3}\nabla^2 Q^{(1)}\,,\\ \nabla^2 Q^{(1)} &= \frac{9}{2}\mathcal{H}^2\frac{\partial_i\partial^j}{\nabla^2}\Pi^{(1)i}_{T\ \ j}\,,\end{aligned} \tag{B.5}$$

where the total anisotropic stress tensor is

$$\Pi^i_{T\ j} = \frac{\bar\rho_\gamma}{\bar\rho_T}\,\Pi^i_{\gamma\ j} + \frac{\bar\rho_\nu}{\bar\rho_T}\,\Pi^i_{\nu\ j}\,. \tag{B.6}$$

We may safely neglect the quadrupole and solve Eq. (B.5) setting $u_\pm = \Phi^{(1)}_\pm\eta$. Then Eq. (B.5), in Fourier space, becomes

$$u'' + \frac{2}{\eta}u' + \left(\frac{k^2}{3} - \frac{2}{\eta^2}\right)u = 0\,. \tag{B.7}$$

This equation has as independent solutions $u_+ = j_1(k\eta/\sqrt{3})$, the spherical Bessel function of order 1, and $u_- = n_1(k\eta/\sqrt{3})$, the spherical Neumann function of order 1. The latter blows up as η gets small and we discard it on the basis of initial conditions. The final solution is therefore

$$\Phi_k^{(1)} = 3\Phi^{(1)}(0)\frac{\sin(k\eta/\sqrt{3}) - (k\eta/\sqrt{3})\cos(k\eta/\sqrt{3})}{(k\eta/\sqrt{3})^3} \tag{B.8}$$

where $\Phi^{(1)}(0)$ represents the initial condition deep in the radiation era.

At the same order in perturbation theory, the radiation velocity can be read off from Eq. (A.4)

$$v_\gamma^{(1)i} = -\frac{1}{2\mathcal{H}^2}\frac{\left(a\partial^i\Phi^{(1)}\right)'}{a}\,. \tag{B.9}$$

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